Risk and Return in Segmented Markets with Expertise*

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Abstract

This paper asks why complex assets and complex investment strategies can earn sustained excess returns despite free entry. In our model, expertise varies across investors and can act as a barrier to entry since demand from investors with higher expertise lowers equilibrium returns and makes participation less attractive to lower expertise investors. Thus, markets are endogenously segmented. Entry determines the fraction of investors who will particpate. Then, participants' portfolio decisions and realized returns determine the joint distribution of financial expertise and financial wealth. Together, along with the deep preference and technology parameters, these distributions determine the aggregate risk bearing capacity and associated required returns to the complex asset. We study how parameters which describe the complexity of the asset naturally determine the market-wide risk return tradeoff, and we show how this risk return tradeoff varies for agents with different levels of financial expertise.

Key Words: segmented markets, slow moving capital, risky arbitrage, hedge funds, industry equilibrium, firm size distribution, financial expertise, intellectual capital, intermediary asset pricing.

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1 Introduction

Complex investment strategies, such as those employed by hedge funds and other sophisticated investors, require investments in specific expertise. These investments can include the development of highly skilled human capital, such as trained financial engineers or computer scientists, the acquisition or construction of financial data, models, and information technology, as well as investments in relationships with brokers, dealers, or in other reputation capital such as a stable investor base. Indeed, such specialized capital may serve as the barrier to entry which potentially enables sophisticated investors to earn higher returns.

This paper aims to understand the role of the stock of intangible capital we call "expertise" in determining equilibrium asset prices. In particular, we provide an explanation of why complex assets and complex investment strategies can earn sustained excess returns despite free entry. In our model, expertise varies across investors and can act as a barrier to entry since demand from investors with higher expertise lowers equilibrium returns and makes participation less attractive to lower expertise investors. Thus, markets are endogenously segmented. Entry determines the fraction of investors who will particpate. Then, participants' portfolio decisions and realized returns determine the joint distribution of financial expertise and financial wealth. Together, along with the deep preference and technology parameters, these distributions determine the aggregate risk bearing capacity and associated required returns to the complex asset. This is because the level and volatility of realized, market level, returns depend on the participation and portfolio choice decisions by investors with heterogeneous amounts of expertise and financial wealth, and the distribution of investor types. We study how parameters which describe the complexity of the asset naturally determine the market-wide risk return tradeoff, and we show how this risk return tradeoff varies for agents with different levels of financial expertise.

We develop an "industry equilibrium" model of investors who acquire expertise and invest in an endogenously segmented market in which the risky asset earns positive excess returns in equilibrium. There is a continuum of agents that choose to be either non-experts, which can invest only in the risk free asset, or experts, which can invest in both the risk free, and risky assets. Entering investors make an initial investment in expertise. This expertise represents the investor's personnel, data, hedging and risk management technologies, back office operations and trade clearing processes, relationships with dealers, and relationships with clients. Thus, participation in the market for the complex risky asset requires a joint investment in a technology which is specific to each investor. In our model, all investors in the market earn a common return in equilibrium, determined by market clearing. However, because each investor's initial joint investment will result in a slightly different overall investment technology, their returns will be subject to idiosyncratic shocks. To fix ideas, one can think of complex asset classes as requiring a joint investment in the asset, and in a hedging technology which requires financial expertise. Importantly, even given the same initial investment, individual investors' expertise outcomes vary, leading to cross sectional variation in the risk/return tradeoff for the complex asset even within the class of experts.

In the stationary equilibrium, there is no aggregate risk, however idiosyncratic risk is priced, as discussed in Merton [1987]. Expertise is used by experts to shrink the volatility of the returns to the risky asset they receive, implying that more sophisticated experts face a higher Sharpe ratio.[1] Thus, expertise may be interpreted as a better ability to hedge risks either by developing a superior model or gathering superior information. Broadly interpreted, these risks may come either from the asset side, or from the liability or fund flow side through investments in a stable investor base. We abstract from the microfoundations of risks from the liability side of funds' balance sheets, and model risk on the asset side.

Two things are important to note about our model of the pricing of complex assets in segmented markets. First, as noted above, the risk in our stationary model is idiosyncratic. This is, of course, a useful assumption technically. However, we argue that it is also realistic, as argued in Merton [1987]. There is a growing literature of the importance of idiosyncratic risk in asset pricing, which we review below. We argue that idiosyncratic risk is likely to be particularly important in markets for complex assets. Complex assets expose their owners to idiosyncratic risk through several channels. First, their constituents tend to be significantly heterogeneous, so that no two investors hold exactly the same asset. Second, the risk management of complex assets typically requires a hedging strategy that will be subject to the individual technological constraints of the investor. Third, firms which manage complex assets may be exposed to key person risk due to the importance of specialized traders, risk managers, and marketers. Finally, complex assets may introduce or amplify idiosyncratic risk on the liability side of the balance sheet, through the fact that they are difficult for outside investors to understand, but tend to be funded with external finance. We use the example of hedge fund investments in mortgage backed securities (MBS), and mortgage backed derivatives. Not only do Gabaix et al. [2007] document a negative relationship between aggregate consumption and prepayments, but assets

[1] See Sharpe [1966].

within the MBS space also have very heterogeneous characteristics and exposures to risk factors such as interest rates and prepayments.[2]

The second, related important assumption in our model is that funds cannot be reallocated across individual investors. Clearly, since the risk in our economy is idiosyncratic, pooling this risk would eliminate the risk premium required to hold it. We rely on the fact that complex assets tend to be held in managed accounts, along with the fact that these managers cannot hedge their own exposure to their particular portfolio. Panageas and Westerfield [2009] and Drechsler [2014] provide important results for the portfolio choice of hedge fund managers who earn fees based on assets under management and portfolio performance. In particular, they show that such agents, in fact, behave like constant relative risk aversion investors. This motivates our choice to model the industry for investing in complex assets as consisting of individual investors with CRRA utility, and further lends support for frictions inhibiting reallocation of funds across managers who are exposed to their own portfolio in order to relax moral hazard constraints.

We present two closely related models to highlight the economic mechanisms driving our results. First, we discuss simple a static model, which we solve fully in closed form, taking the joint distribution of wealth and expertise as given. Taking the joint distribution as given allows us to do comparative statics on the parameters that describe this distribution. In particular, we provide results for the effects of changes in this distribution, and the other model parameters, on the market clearing excess return to the complex asset, individual Sharpe ratios, and the equilibrium weighted average market Sharpe ratio. We emphasize the heterogeneity in Sharpe ratios, and the difference between individual Sharpe ratios and the market-wide risk return trade off. For example, we show that if fundamental volatility increases, there is a cutoff level of expertise below which individual Sharpe ratios decrease, and above which they increase. This result is interesting in the context of changes in participation following shocks to the complex asset market. The assumption that risk is idiosyncratic is semantic in the static model, and thus this model can be interpreted to have aggregate risk.

Next, we present a dynamic model in continuous time. In this model, expertise varies in the cross-section but is fixed for each agent over time. We also solve this model up to the equilibrium fixed point for expected returns in closed form, including the joint wealth

[2]MBS have been described as the Swiss Army knife of asset classes, providing any risk exposure one desires. Likewise, the market for mortgage backed derivatives has been described as analogous the heterogeneous detritus left over from butchering a pig after the desirable parts have satisfied the demands of long-only investors.

and expertise distribution.[3] In this model, the deep preference and technology parameters determine the joint distribution of wealth and expertise in the long run, stationary equilibrium. The dynamic model endoegenizes the effect of investors' consumption/savings and portfolio choices on this joint distribution. We show that the wealth distribution is Pareto conditional on each expertise level. The decay parameter depends on investors' portfolio choice and exposure to the risky complex asset. In particular, because investors with higher expertise choose a higher exposure to the risky asset, both the drift and the volatility of their wealth will be greater, leading to a fatter tailed distribution at higher expertise levels. We use our results for how expertise level specific wealth distributions are determined, along with results describing expertise level and aggregate demand for the risky asset, to show how changes in the deep parameters of the model lead to changes in equilibrium returns which are consistent with more complex assets having higher expected returns. We also present comparative statics over equilibrium individual Sharpe ratios, and the equilibrium weighted average market Sharpe ratio.

The paper proceeds as follows. In Section 2 we review the related literature. We present and analyze our static model in Section 3. Section 4 contains the construction and analysis of our dynamic model, and finally Section 5 concludes. Most proofs appear in the Appendix. In separate work (Eisfeldt et al. [2015]), we study a discrete time dynamic model with stochastic expertise, which we use to study the impact of unanticipated aggregate shocks and to develop quantitative results. In particular, using intuition developed in this paper, we show that expertise can act as a barrier to entry and to interesting dynamics for market excess returns and volatility following shocks to investor wealth and to fundamental asset volatility.

2 Literature

Our paper contributes to a large and growing literature on segmented markets and asset pricing. We group the existing literature into three main categories, namely financial constraints and limits to arbitrage, intermediary asset pricing, and segmented market models with alternative microfoundations to agency theory. Although our model is not one of arbitrage per se, our study shares the goal of understanding the returns to complex assets and strategies. Our model also shares the features of segmented markets and trading frictions with the limits to arbitrage literature. Gromb and Vayanos [2010b] provide a recent survey of the theoretical lit-

[3] We use a numerical solution for market clearing. However, the solution is straightforward given our analytical solution for policy functions and distribution over individual states.

erature on limits to arbitrage, starting with the early work by Brennan and Schwartz [1990] and Shleifer and Vishny [1997].[4] Shleifer and Vishny [1997] emphasize that arbitrage is conducted by a fraction of investors with specialized knowledge, but similar to He and Krishnamurthy [2012], they focus on the effects of the agency frictions between arbitrageurs and their capital providers. Although we do not explicitly model risks to the liability side of investors' balance sheets, one can interpret the shocks agents in our model face to include idiosyncratic redemptions.[5] Recently, the broader asset pricing impact of financially constrained intermediaries has been studied in the literature on intermediary asset pricing following He and Krishnamurthy [2012] and He and Krishnamurthy [2013].[6] This literature applies results from the literature on asset pricing with heterogenous agents, following Dumas [1989], to segmented markets with financial constraints.[7] In doing so, the intermediary asset pricing literature both connects to empirical applications, and to the asset price dynamics which are the focus of the limits to arbitrage literature. Finally, several papers develop alternative microfoundations to agency theory for segmented markets. Plantin [2009] develops a model of learning by holding. Duffie and Strulovici [2012] develop a theory of capital mobility and asset pricing using search foundations. Glode, Green, and Lowery [2012] study asset price dynamics in a model of financial expertise as an arms race in the presence of adverse selection. Kurlat [2013] studies an economy with adverse selection in which buyers vary in their ability to evaluate the quality of assets on the market, and, like us, emphasizes the distribution of expertise on the equilibrium price of the asset. Garleanu, Panageas, and Yu [2014] derive market segmentation endogenously from differences in participation costs. Kacperczyk, Nosal, and Stevens [2014] construct a model of consumer wealth inequality from differences in investor sophistication.

Our model is an example of an "industry equilibrium" model in the spirit of Hopenhayn [1992a] and Hopenhayn [1992b]. These models study the important effects of firm dynamics, entry and exit in the heterogeneous agent framework developed in Bewley [1986]. This liter-

[4]See also Aiyagari and Gertler [1999], Froot and O'Connell [1999], Basak and Croitoru [2000], Xiong [2001], Gromb and Vayanos [2002], Yuan [2005], Gabaix, Krishnamurthy, and Vigneron [2007], Mitchell, Pedersen, and Pulvino [2007], Acharya, Shin, and Yorulmazer [2009], Kondor [2009], Duffie [2010], Gromb and Vayanos [2010a], Hombert and Thesmar [2011], Edmond and Weill [2012], Mitchell and Pulvino [2012], Pasquariello [2013], and Kondor and Vayanos [2014].

[5]For other models of risks stemming from redemptions and fund outflows and the resulting asset pricing implications, see Berk and Green [2004], and Liu and Mello [2011].

[6]See also, for example, Adrian and Boyarchenko [2013]. For empirical applications, see for example, Adrian, Etula, and Muir [Forthcoming] and Muir [2014].

[7]For closely related work on asset pricing with heterogeneous risk aversion and segmented markets, see also Basak and Cuoco [1998], Kogan and Uppal [2001], Chien, Cole, and Lustig [2011], and Chien, Cole, and Lustig [forthcoming].

ature focuses in large part on explaining firm growth, and moments describing the firm size distribution. Recent progress in the firm dynamics literature using continuous time techniques to solve for policy functions and stationary distributions include Miao [2005], Luttmer [2007], Gourio and Roys [2014], Moll [Forthcoming], and Achdou, Han, Lasry, Lions, and Moll [2014]. We draw on intuition from these papers as well as discrete time models of firm dynamics, as in recent work by Clementi and Palazzo [2014], which emphasizes the role of selection in explaining the observed relationships between firm age, size, and productivity. We also build on work from the city size literature in Gabaix [1999] and the literature on the consumer wealth distribution with idiosyncratic risk and fiscal policy in Benhabib et al. [2014].

Idiosyncratic risk is priced in our model, as classically developed in Merton [1987], and more recently studied in Malkiel and Xu [2006]. Although Ang et al. [2006] and Ang et al. [2009] argue that idiosyncratic risk is negatively related to excess returns in the cross-section, Fu [2009] argues that this seemingly counter-intuitive result may be due to return reversal in a subset of small stocks with high idiosyncratic volatilities. That paper further shows that returns are in fact positively related to expected idiosyncratic risk constructed at the firm level using a GARCH model. Corroborative results controlling for liquidity effects appear in Spiegel and Wang [2005]. Goyal and Santa-Clara [2003] present related evidence of aggregate return predictability from idiosyncratic stock return risk. Finally, Pontiff [2006] investigates the role of idiosyncratic risk faced by arbitrageurs in several stylized models and in a review of the literature and argues that "The literature demonstrates that idiosyncratic risk is the single largest cost faced by arbitrageurs".

We use the asset backed fixed income (ABFI) segment of the hedge fund industry for empirical moments describing size and performance. As such, we draw from the literature on hedge funds performance and compensation.[8] In particular, we motivate our use of ABFI funds as our main example of a complex strategy using the evidence in Duarte, Longstaff, and Yu [2006]. They provide evidence that MBS strategies are relatively complex and earn higher returns even in comparison to other sophisticated fixed income arbitrage strategies. Several papers provide evidence for the importance of idiosyncratic risk in the hedge fund returns, following the idea in Merton [1987] that idiosyncratic risk will be priced when there

[8] Fung, Hsieh, Naik, and Ramadorai [2008] is a well known paper describing performance. Jagannathan, Malakhov, and Novikov [2010] carefully correct for selection bias and smoothed returns in a study of hedge fund performance persistence. Carlson and Steinman [2007] consider the relationship between hedge fund survival and market conditions. In a related spirit to our work, Getmansky [2012] empirically studies the effects of size and competition on hedge fund returns.

are costs associated with learning about or hedging a specific asset.[9] Relatedly, Fung and Hsieh [1997] find that hedge fund returns have low and sometimes negative correlation with asset class returns. Our model features investors with constant relative risk aversion (CRRA) preferences. While we do this for tractability and parsimony to retain our focus on the effects of the joint wealth and expertise distribution, Panageas and Westerfield [2009] show that hedge fund compensation contracts with long horizons lead to portfolio choice which aligns perfectly with that of a CRRA investor. Drechsler [2014] extends these results to include variation in managers' outside options and shows the CRRA result holds as long as such reservation values are neither too high nor too low. These results extend the analysis of the impact of high-water marks in Goetzmann et al. [2003].

The majority of the assets under management in the ABFI sector are mortgage backed securities (MBS). Cash flow risk in MBS securities typically comes primarily from prepayment risk, since the largest part of the market consists of agency securities. Gabaix, Krishnamurthy, and Vigneron [2007] provide convincing evidence that returns are driven in large part by limits to arbitrage. Importantly, they show that although prepayment risk is partly common within a class of MBS securities, the risk in MBS investing is negatively correlated with the aggregate risks born by a representative consumer. Recent work by Boyarchenko, Fuster, and Lucca [2014] extends these ideas and provides evidence that prepayment model risk explains the "smile" in MBS option adjusted spreads (OAS) and confirms that time series variation in returns is closely related to the MBS supply relative to the capital of MBS investors.[10] That idiosyncratic risk is priced in MBS is supported by prior empirical studies. It is also consistent with the fact that different investors have different pricing and hedging models, and invest in different parts of the mortgage space. Some funds may benefit from early prepayment, while other funds may are suffer from early prepayment. Different mortgage assets have different direct interest rate exposure, and investors hedge their interest rate exposures to different extents. Finally, variation in lending standards can have opposite effects on prepayment due to default and voluntary prepayment. We implement the "model risk" inherent in funds' prepayment models via the variation in idiosyncratic risk faced by investors with varying amounts of expertise.

[9]See Titman and Tiu [2011] and Lee and Kim [2014]. Jurek and Stafford [Forthcoming] emphasize that scarce and specialized knowledge may drive both hedge fund returns and put pricing.

[10]See also Dunn and McConnell [1981a], Dunn and McConnell [1981b], Schwartz and Torous [1992], Stanton [1995], Boudoukh, Richardson, Stanton, and Whitelaw [1997], Longstaff [2005], Downing, Stanton, and Wallace [2005] for models of MBS pricing, and Agarwal, Driscoll, and Laibson [2013] for a recent model of consumer prepayment behavior solved in closed form.

3 Static Model

We present a static model to build intuition about the interaction between the size and expertise distribution of investors and equilibrium returns.

Setup Investors have constant relative risk aversion preferences over date 1 consumption, with coefficient of relative risk aversion γ. At date 0, they are endowed with financial wealth W and expertise X. There is a riskless asset with gross return R_f, and a risky asset with gross returns R, which are distributed log normally. We use lower case letters to denote logs.

We assume that the log return on the risky asset for any given investor, which we denote by r, is distributed according to $r \sim N(\boldsymbol{\mu} - \frac{1}{2}\frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X})$, given the distribution of W and X. We denote the variance of log returns on the fundamental asset, before expertise is applied, by σ_v^2, and call this fundamental variance, and its square root fundamental volatility. The effective variance and volatility of an investor's return on the risky asset then decreases as expertise X increases, while the innovation v itself is independent from W and X.

Investing in the complex asset implies a joint investment in a common market clearing return, as well as a specific risk from hedging or asset specificities. We assume the specified functional form for log return volatility for simplicity, as it allows for straightforward calculations of all expectations, and minimal parameters. It is straightforward to show that our main conclusions for the static model are robust to a family of functions $\frac{\sigma_v^2}{k_0+k_1X+k_2X^2+\ldots}$, with all coefficients $k_0, k_1, k_2....$ being non-negative. In levels, expected returns $\boldsymbol{\mu}$ are the same for all investors, regardless of their individual expertise.

Solving the Portfolio Choice Problem Using the approximation described in Campbell and Viceira [2002a], and the associated appendix Campbell and Viceira [2002b], which relates log individual-asset returns to log portfolio returns over short time intervals, the investor's optimization problem becomes:

$$\max_{\theta} \left\{ \theta \left(\boldsymbol{\mu} - r_f \right) - \frac{\gamma}{2} \theta^2 \frac{\sigma_v^2}{X} \right\} \tag{1}$$

where r_f represents the log return on the riskless asset. In this section, for emphasis, we use bold notation to denote equilibrium returns.[11] The solution for the optimal fraction of wealth

[11]Because an individual investor's return volatility depends on their expertise, for the approximation to be good given our specification for log return volatility, we have to impose a technical restriction that the majority of distribution of expertise X is bounded away from zero. This assumption is unnecessary if one adopts the

allocated to the risky asset is:

$$\theta^* = \frac{(\boldsymbol{\mu} - r_f)}{\gamma \sigma_v^2} X. \tag{2}$$

Thus, portfolio choice in a lognormal model with power utility resembles that of a mean variance investor. The allocation to the risky asset is increasing in the equilibrium average excess return, decreasing in risk aversion, and decreasing in the fundamental shock variance. Moreover, the fraction of wealth that an investor allocates to the risky asset strictly increases with expertise. The relationship is linear under our functional form assumptions.[12]

Equilibrium We now describe how the equilibrium excess return depends on the parameters for preferences, technology, and the joint distribution of wealth and expertise. We focus on comparative statics over the equilibrium average excess return, market level Sharpe ratio, and individual Sharpe ratios. We normalize the mass of investors to one, define the value of the supply of the risky asset to be S, determine the market clearing log expected return $\boldsymbol{\mu}$, and then back out the equilibrium expected level return and therefore $\boldsymbol{\alpha}$. We assume that $\log(W)$ and $\log(X)$ are jointly normally distributed. We denote the joint pdf of the log variables $f(w, x)$, with means and variances μ_w, μ_x, σ_w^2, and σ_x^2 respectively, and covariance $\rho_{w,x}\sigma_w\sigma_x$. Thus, an economy ψ is described by $\psi \equiv \{r_f, \gamma, I, \sigma_v, \mu_w, \sigma_w, \mu_x, \sigma_x, \rho_{w,x}\}$. The equilibrium log expected return $\boldsymbol{\mu}$ solves the market clearing condition:

$$\text{Supply} \equiv S = \text{Demand} = \int\int \exp(w)\theta^*\left(\exp(x)\right) f(w, x)\, dw\, dx = \frac{\boldsymbol{\mu} - r_f}{\gamma \sigma_v^2} \boldsymbol{X} \tag{3}$$

where $\theta^*\left(\exp(x)\right)$ is the portfolio choice given in Equation (23) and $\boldsymbol{X}$ is the wealth and population weighted average of expertise:

$$\int\int \exp(w + x) f(w, x)\, dw\, dx = \exp\left(\frac{1}{2}\left(\sigma_w^2 + \sigma_x^2 + 2\rho_{w,x}\sigma_w\sigma_x + 2\mu_w + 2\mu_x\right)\right) \tag{4}$$

utilizing the result for the expectation of log normally distributed variables.

Rearranging, we have:

$$\boldsymbol{\mu} = r_f + \left(\frac{\sigma_v^2}{\boldsymbol{X}}\right) \gamma S. \tag{5}$$

The equilibrium log expected excess return is increasing in the amount of risk relative to the

general functional form for volatility discussed in footnote 3.

[12]Without restrictions on the distribution of X, θ can be larger than one, implying borrowing at the risk free rate.

risk bearing capacity of investors. We decompose the inputs into two components. The first term is the effective risk in the market, namely the fundamental risk σ_υ^2, scaled down by the wealth and population weighted average of expertise. The second term is the risk aversion scaled supply of the risky asset which must be cleared. The higher is investors' coefficient of relative risk aversion, and the larger is the supply of the asset, the higher is the required return. Conversely, the wealth and population weighted average of expertise, $\boldsymbol{X}$, scales $\boldsymbol{\mu}$ down due to the positive impact of expertise on investors' allocation to the risky asset.

Using the equilibrium log expected return $\boldsymbol{\mu}$, we can rewrite agents' optimal portfolio allocations to the risky asset as:

$$\theta^* = \frac{X}{\boldsymbol{X}} S. \tag{6}$$

This expression captures the fact that, in equilibrium, the optimal portfolio allocations to the risky asset by an agent with expertise X turns out to be a fraction of total supply equal to their expertise relative to the wealth and population weighted average of expertise.

The equilibrium mean of the level of the gross risky return over the level of the gross risk free rate, $\boldsymbol{\alpha}$, is a monotonic transformation of $\boldsymbol{\mu}$. In particular, we show in the Appendix that the equilibrium α is then given by:

$$\boldsymbol{\alpha} = \exp\left(\boldsymbol{\mu}\right) - R_f \tag{7}$$

which gives a one to one mapping from μ to α conditional on parameters. Note also that writing θ^* (Equation 23) as a function of either $\boldsymbol{\mu}$ or $\boldsymbol{\alpha}$ will always yield identical equilibrium outcomes.

Lemma 1 *Using Equation (7) describing the equilibrium market clearing $\boldsymbol{\alpha}$, the following comparative statics can be directly calculated:*

1. $\frac{\partial \boldsymbol{\alpha}}{\partial \sigma_\upsilon^2} = \exp\left(\boldsymbol{\mu}\right) \frac{\gamma S}{\boldsymbol{X}} > 0$. *$\boldsymbol{\alpha}$ increases with fundamental risk.*

2. $\frac{\partial \boldsymbol{\alpha}}{\partial \gamma} = \exp\left(\boldsymbol{\mu}\right) \frac{\sigma_\upsilon^2}{\boldsymbol{X}} S > 0$. *$\boldsymbol{\alpha}$ increases with the coefficient of relative risk aversion.*

3. $\frac{\partial \boldsymbol{\alpha}}{\partial S} = \exp\left(\boldsymbol{\mu}\right) \frac{\sigma_\upsilon^2}{\boldsymbol{X}} \gamma > 0$. *$\boldsymbol{\alpha}$ increases with the supply of the risky asset investors must absorb.*

4. $\frac{\partial \boldsymbol{\alpha}}{\partial \mu_w} = -\exp\left(\boldsymbol{\mu}\right) \frac{\sigma_\upsilon^2}{\boldsymbol{X}} \gamma S < 0$. *$\boldsymbol{\alpha}$ decreases as aggregate wealth increases.*

5. $\frac{\partial \boldsymbol{\alpha}}{\partial \mu_x} = -\exp\left(\boldsymbol{\mu}\right) \frac{\sigma_\upsilon^2}{\boldsymbol{X}} \gamma S < 0$. *$\boldsymbol{\alpha}$ decreases as aggregate expertise increases.*

6. $\frac{\partial \boldsymbol{\alpha}}{\partial \rho_{w,x}} = -\exp\left(\boldsymbol{\mu}\right)\frac{\sigma_v^2}{\boldsymbol{X}}\gamma S \sigma_w \sigma_x < 0$. *As* $\rho_{w,x}$ *increases, there is a more efficient allocation of expertise and* $\boldsymbol{\alpha}$ *decreases.*

7. $\frac{\partial \boldsymbol{\alpha}}{\partial \sigma_w} = -\exp\left(\boldsymbol{\mu}\right)\frac{\sigma_v^2}{\boldsymbol{X}}\gamma S\left(\sigma_w + \rho_{w,x}\sigma_x\right)$

   - > 0 *if* $\rho_{w,x} < -\frac{\sigma_w}{\sigma_x}$, *i.e. if wealth and expertise are strongly negatively correlated.*
   - < 0 *if* $\rho_{w,x} > -\frac{\sigma_w}{\sigma_x}$, *i.e. if wealth and expertise are positively or only weakly negatively correlated.*

8. $\frac{\partial \boldsymbol{\alpha}}{\partial \sigma_x} = -\exp\left(\boldsymbol{\mu}\right)\frac{\sigma_v^2}{\boldsymbol{X}}\gamma S\left(\sigma_x + \rho_{w,x}\sigma_w\right)$

   - > 0 *if* $\rho_{w,x} < -\frac{\sigma_x}{\sigma_w}$, *i.e. if wealth and expertise are strongly negatively correlated.*
   - < 0 *if* $\rho_{w,x} > -\frac{\sigma_x}{\sigma_w}$, *i.e. if wealth and expertise are positively or only weakly negatively correlated.*

Proof. By direct calculation. ■

All comparative statics are intuitive. An increase in the correlation of wealth and expertise will reduce $\boldsymbol{\alpha}$, as investors with more expertise account for a larger share of the wealth distribution. The effect of an increase in $\rho_{w,x}$ on the market clearing $\boldsymbol{\alpha}$ will be larger the larger is amount of fundamental risk, σ_v^2, the larger is the coefficient of relative risk aversion, γ, the larger is the supply of the risky asset, S, the smaller is the mean of log wealth, μ_w, and the smaller is the mean of log expertise, μ_x.

We also derive results for the equilibrium market-level and investor-specific Sharpe ratios. The market level Sharpe ratiorequires a definition appropriate for our environment. Here, we define the equilibrium market level Sharpe ratio to be the equal-weighted cross-sectional average of excess returns divided by the equal-weighted cross-sectional standard deviation. Thus this market Sharpe ratio can, for example, be interpreted as the expected Sharpe ratio for an investor "behind the veil" drawing from the distribution of possible levels of expertise, before the investment stage of the model. This Sharpe ratio would be relevant, for example, in a model with entry in which an investor must decide whether to enter before drawing an expertise level from the given distribution. We then refer to what is technically the equilibrium equally weighted market Sharpe ratio as the "Sharpe ratio" for exposition purpose:

$$SR = \frac{1 - R_f \, \exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X}\right)\right] - 1}} = \frac{1 - R_f \exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\sum_{k=1}^{\infty} \frac{1}{k!}\sigma_v^{2k} \exp(-k\mu_x + \frac{1}{2}k^2\sigma_x^2)}} \approx \frac{1 - R_f \exp\left(-\boldsymbol{\mu}\right)}{\sigma_v \exp(-\frac{1}{2}\mu_x + \frac{1}{4}\sigma_x^2)}, \quad (8)$$

where $\mathbb{E}$ denotes the cross-sectional expectation. This ratio aggregates all investor decisions and measures the *market level* risk return tradeoff.[13] The market-level Sharpe ratio increases as the average log expertise μ_x in this economy increases, but it decreases as the cross-sectional standard deviation of log expertise σ_x^2 increases.

Lemma 2 *Using Equation (8) describing the equilibrium market clearing equally weighted Sharpe ratio, the following comparative statics can be directly calculated:*

1. *Let η denote any parameter $\eta \in \{\gamma, S, \mu_w, \sigma_w, \rho_{w,x}\}$.*
   Then, $Sign\left(\frac{\partial(SR)}{\partial\eta}\right) = Sign\left(\frac{\partial\boldsymbol{\alpha}}{\partial\eta}\right)$.
2. *The signs for comparative statics with respect to parameters $\hat{\eta} \in \{\sigma_v^2, \mu_x, \sigma_x\}$, are indeterminate.*

Proof. By direct calculation, see Appendix. ■

Expected returns rise proportionally relative to the volatility of the risky asset return in our static model, so that the Sharpe ratio improves with any parameter change that increases $\boldsymbol{\alpha}$. Thus, we confirm that, at the market level, parameter changes which lead to an increase in the equilibrium expected excess return in fact lead to better investment opportunities given the market risk in equilibrium.

In our model, each investor confronts a different risk-return trade-off. Since the volatility of log returns depends on an individual investors' expertise, an observed increase in the market Sharpe ratio does not necessarily guarantee a higher Sharpe ratio for every investor in the market. Moreover, even if the Sharpe ratio improves for each agent individually, the magnitude of the improvement an individual investor faces will not, in general, coincide with the market improvement. To see this, consider the investor-specific Sharpe ratio. For an investor with wealth W and expertise X, we show in the Appendix that this investor's Sharpe ratio is given by:

[13] See Appendix for derivation. We also compute and analyze the market value weighted Sharpe ratio in the Appendix.

$$SR\left(W,X\right)=\frac{1-R_{f}\,\exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\exp\left(\frac{\sigma_{v}^{2}}{X}\right)-1}}. \tag{9}$$

Equation (9) clearly shows that the model can deliver considerable cross-sectional dispersion in investor-specific Sharpe ratios.[14] Investors with very low effective risk, $\frac{\sigma_v^2}{X}$, face significantly higher Sharpe ratios than their counterparts with low expertise. We can determine the signs of the following comparative statics:

Lemma 3 *Using Equation (9) describing the investor-specific Sharpe ratio, and Equation (23) describing the portfolio allocation θ^*, the following comparative statics can be directly calculated. Let η denote any parameter $\eta \in \{\gamma, S, \mu_w, \sigma_w, \rho_{w,x}\}$.*[15]

1. $\frac{\partial SR(W,X)}{\partial X} > 0$. *Higher expertise generates lower effective risk, and a correspondingly higher individual Sharpe ratio.*

2. $Sign\left(\frac{\partial SR(W,X)}{\partial \eta}\right) = Sign\left(\frac{\partial \boldsymbol{\alpha}}{\partial \eta}\right) = Sign\left(\frac{\partial (SR)}{\partial \eta}\right)$. *All investor-specific Sharpe ratios co-move with the equilibrium excess return and the market level equilibrium Sharpe ratio.*

3. $Sign\left(\frac{\partial \boldsymbol{Var}(SR(W,X))}{\partial \eta}\right) = Sign\left(\frac{\partial \boldsymbol{\alpha}}{\partial \eta}\right) = Sign\left(\frac{\partial (SR)}{\partial \eta}\right)$. *Whenever a parameter change increases the market level equilibrium Sharpe ratio, it leads to a larger cross-sectional dispersion in the investor-specific Sharpe ratio.*

4. $Sign\left(\frac{\partial^2 SR(W,X)}{\partial \eta \partial X}\right) = Sign\left(\frac{\partial SR(W,X)}{\partial \eta}\right) = Sign\left(\frac{\partial \boldsymbol{\alpha}}{\partial \eta}\right) = Sign\left(\frac{\partial (SR)}{\partial \eta}\right)$.
Whenever a parameter change increases the investor-specific Sharpe ratio, it leads to a larger increase for high expertise investors relative to low expertise investors.

5. $Sign\left(\frac{\partial^2 \theta^*(W,X)}{\partial \eta \partial X}\right) = Sign\left(\frac{\partial SR(W,X)}{\partial \eta}\right) = Sign\left(\frac{\partial \boldsymbol{\alpha}}{\partial \eta}\right) = Sign\left(\frac{\partial (SR)}{\partial \eta}\right)$.[16]
Whenever a parameter change increases the investor-specific portfolio allocation, it leads to a larger increase for high expertise investors relative to low expertise investors.

6. $\exists \bar{X} > \underline{X} > 0$ *such that* $\forall X > \bar{X}$, $\frac{\partial SR(W,X)}{\partial \sigma_v^2} > 0$, *and* $\forall X < \underline{X}$, $\frac{\partial SR(W,X)}{\partial \sigma_v^2} < 0$. *An increase in the fundamental risk generates a higher Sharpe ratio for high expertise investors and a lower Sharpe ratio for low expertise investors.*

[14] Note this would be the case even if one adopted a flexible functional form for effective risk.

[15] Derivatives with respect to μ_x and σ_x follow the same formulas as those that support parts 2 to 5 of lemma 3. However, the changes are not comparable to the market Sharpe ratio, as we can't determine the signs in lemma 2, part 2. Derivatives with respect to σ_v^2 cannot be signed generally.

[16] Except for γ, where $\frac{\partial^2 \theta^*(W,X)}{\partial \gamma \partial X} = 0$.

Proof. By direct calculation. See Appendix. ■

Lemma (3) has rich implications. First, we emphasize the co-movement between cross sectional variation in investor-specific Sharpe ratios and the level of the market Sharpe ratio. Any increase in the market-level Sharpe ratio will also increase the cross-sectional dispersion in Sharpe ratios. Furthermore, because an increase in the market level Sharpe ratio improves investment opportunities for high expertise investors by more than for low expertise investors, such an increase accordingly increases their allocation to the risky asset θ^* by more. Thus, an improvement in the market-level risk return tradeoff in large part reflects the improved risk-return trade-off faced by high expertise investors, and not by their low expertise counterparts.

Any parameter change which increases the market level Sharpe ratio increases the investor specific Sharpe ratio for high expertise by more, and increases the influence of high expertise investors' Sharpe ratios on the market level risk return tradeoff. In our model, measured improvements in the aggregate Sharpe ratio are a misleading indicator of improvements in individual investors' risk-return tradeoff, and can indeed more accurately reflect changes in the Sharpe ratio of higher expertise investors. The converse is also true.

Furthermore, part 6 of Lemma (3) states that changes to fundamental risk can lead to changes in individual Sharpe ratios that vary in sign. For example, if σ_v^2 increases, all investors face the same increase in the equilibrium excess return, but investors with high expertise face a considerably smaller increase in risk. We also emphasize that because an increase in fundamental risk improves the investor-specific Sharpe ratio for some investors but not others, in a dynamic model a shock to fundamental risk can lead potentially lead to variation in investors' participation decisions. In other words, an increase in risk which improves the market level equally weighted Sharpe ratio may still lead low expertise investors to exit, or not to enter.

4 Dynamic Model

4.1 Preferences, Endowments, & Technologies

We study a model with a continuum of investors of measure one, with CRRA utility functions over consumption:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

Investors are endowed with a level of expertise which varies in the cross section, but is fixed for each agent over time. They have access to a riskless technology, as well as a risky technology with a variance that is decreasing in the investor's level of expertise. Each individual investor is born with a fixed expertise level, x, drawn from a distribution with pdf $\lambda(x)$, and cdf $\Lambda(x)$. We require that this distribution satisfies $\lim_{x\to\infty}\sigma(x)=\underline{\sigma}>0$. The lower bound on volatility, $\underline{\sigma}$, represents some complex asset risk that cannot be eliminated even by the agents with the greatest expertise, and it guarantees that the growth rate of wealth is finite. We further define the "highest" level of expertise as $g(\bar{x})\equiv\sup_x g(x)$.

Investors can choose to be experts, and have access to the complex risky asset, or non-experts, who can only invest in the risk free asset. To invest in the complex risky asset, an investor must pay the entry cost F_{nx} to set up their specific technology for investing in the complex risky asset. Experts must also pay a maintenance cost, F_{xx} to maintain continued access to the risky technology. Thus, one can consider that investors are born with a potential level of expertise, which they can achieve, along with access to the complex risky asset, by investing the entry cost F_{nx}. Investors will choose to be experts if and only if the benefit of being an expert, as measured by the excess return on the risky asset relative to the effective variance they face given their expertise, is high enough to compensate them for paying the entry and maintenance costs. We consider the case in which both the entry and maintenance are proportional to wealth:

$$\begin{aligned} F_{nx} &= f_{nx}w, \\ F_{xx} &= f_{xx}w. \end{aligned}$$

which yields value functions which are homogeneous in wealth. In the stationary equilibrium we study, if an investor chooses to enter initially, they will also always choose to maintain their expertise by paying the flow maintenance cost F_{xx}, rather than relinquish their expertise stock along with access to the risky asset. Thus, in this model with fixed expertise and homogeneity in wealth, there is no exit in a stationary equilibrium. Nonetheless, overall participation, and hence aggregate risk bearing capacity, is endogenous.

We first describe the Bellman equations for non-experts and experts respectively, and characterize their value functions, as well as the associated optimal policy functions. With the value functions of experts and non-experts in hand, we then characterize the entry decision.

We begin with non-experts, who can only invest in the risk free asset. Let $w\left(t,s\right)$ denote the wealth of investors at time s with initial wealth W_t at time t. The riskless asset delivers a fixed return of r_f. All investors choose consumption, and an optimal stopping, or entry time according to the Bellman Equation:

$$V^n\left(w\left(t,s\right),x\right) = \max_{c^n(t,s),\tau} E\left[\int_t^\tau e^{-\rho(s-t)}u\left(c^n\left(t,s\right)\right)ds + e^{-\rho(\tau-t)}V^x\left(w\left(t,s\right)-F_{nx},x\right)\right] \quad (10)$$

$$\text{s.t. } dw\left(t,s\right) = \left(r_f w\left(t,s\right)-c^n\left(t,s\right)\right)ds \quad (11)$$

where ρ is their subjective discount factor, $c\left(t,s\right)$ is consumption at time s, F_{nx} is the entry cost, and τ is the optimal entry date. Under the assumptions of linear entry and maintenance costs, and expertise which is fixed over time, the optimal entry date in a stationary equilibrium will be either immediately or never. Thus, assuming an initial stationary equilibrium, investors who choose an infinite stopping time are then non-experts, and investors who choose a stopping time $\tau = t$ are experts.[17]

Experts can invest in a complex risky asset as well as the riskless asset. The complex risky asset delivers a stochastic return which follows a geometric Brownian motion:

$$\frac{dP\left(t,s\right)}{P\left(t,s\right)} = \left[r_f+\alpha\left(s\right)\right]ds+\sigma\left(x\right)dB\left(t,s\right) \quad (12)$$

where $\alpha\left(s\right)$ is the equilibrium excess return on the risky asset which clears the exogenously given supply, and $B\left(t,s\right)$ is a standard Brownian motion. Note that we are abusing notation and suppressing the dependence or returns on the specific investor. We denote the expertise-specific volatility of the returns to the complex risky asset $\sigma\left(x\right)$. In order to invest in the risky asset, and earn its common market clearing return, an investor must also jointly invest in a technology with a zero mean return and an idiosyncratic shock. This technology represents each investor's specific hedging and financing technologies, as well as the unique features of their particular asset. Thus, the sum of the common market clearing return, and the idiosyncratic shock, represents an investor's total returns to the complex risky asset. Investors with greater expertise operate technologies with lower risk, i.e. we assume $\frac{\partial\sigma(x)}{\partial x} < 0$.

By definition, α cannot be generated by bearing systematic risk. Instead, we argue that investing in complex assets exposes investors to idiosyncratic risk. One can think of α in our model as being due to mispricing. However, capturing α due to mispricing is risky because it requires a model. Each investor's model is different, and exposes them to an investor specific (i.i.d.) shock. This is very realistic since hedge ratios tend to vary substantially across different

[17] Outside of the stationary equilibrium, because α is not constant, both entry and exit are possible.

investors in the same asset class.[18] To be concrete, consider the following motivation for the return process in (12). There is a fundamental complex asset, such as an MBS or convertible bond, which returns:

$$\frac{dF(t,s)}{F(t,s)} = [r_f + \alpha(s) + a(s)]\,dt + \sigma^F dB^F(t,s). \tag{13}$$

Note that, σ^F is specific only to the asset, not to the investor. Investors have heterogeneous access to, or knowledge of tracking or hedging portfolios with investor specific returns:

$$\frac{dT_i(t,s)}{T_i(t,s)} = [r_f + a(s)]\,dt + \sigma^F dB^F(t,s) - \sigma(x) dB_i^T(t,s). \tag{14}$$

In order to invest in the complex asset, investors must take a negative position in their tracking portfolio.[19] We do not clear the market for tracking portfolios. We instead argue that it is realistic to assume that demand for the tracking portfolio from hedging the complex asset is "small" relative to total demand. We assume that $\alpha >> a$ and then interpret the α as net of the excess return on the tracking portfolio, i.e. as the mispricing of the complex asset, or equivalently as the return it earns because investors must bear idiosyncratic risk to invest in it. The fact that $dB_i^T(t,s)$ is i.i.d. across investors can be interpreted as each investor using their own model, investing in their own niche of the complex asset market, or as a random trading cost or client funding shock. Any risk that is common across investors is subsumed by $dB^F(t,s)$. One interpretation of $\sigma(x)\,dB_i^T(t,s)$ is as "hedging model risk". Investors with higher x have access to or knowledge of tracking portfolios with a higher correlation, which we implement by assuming that $\frac{\partial \sigma(x)}{\partial x} < 0$. The expected return on the complex asset net of the tracking portfolio is α. The variance of the return on the complex asset net of the tracking portfolio investment is $\sigma(x)$, which is decreasing in expertise. In other words, the net return of going long the fundamental complex asset and short the tracking portfolio is described in equation (12).

Experts allocate their wealth between current consumption, a risky asset, and a riskless asset. They also choose an optimal stopping time to exit the market, which we include for

[18]For example, there is no agreed upon method to hedge mortgage duration risk, though most all active MBS investors do so. Some hedge according to empirical durations, using various estimation periods and rebalancing periods. Others hedge according to the sensitivity of MBS prices yield curve shifts using their own (widely varying) proprietary model of MBS prepayments and prices.

[19]Equivalently, assume a restriction on $\sigma(x)$ relative to σ and a such that this will be optimal.

completeness, however exit will not occur in this homogeneous model with fixed expertise.

$$\begin{aligned} V^x\left(w\left(t,s\right),x\right) &= \max_{c^x(t,s),T,\theta(x,t,s)} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)} u\left(c^x\left(t,s\right)\right) ds + e^{-\rho(T-t)} V^n\left(w\left(t,s\right),x\right)\right] && (15) \\ \text{s.t. } dw\left(t,s\right) &= \left[w\left(t,s\right)\left(r_f + \theta\left(x,t,s\right)\alpha\left(t,s\right)\right) - c^x\left(t,s\right) - F_{xx}\right] ds && (16) \\ & +w\left(t,s\right)\theta\left(x,t,s\right)\sigma\left(x\right) dB\left(t,s\right), \end{aligned}$$

where $\alpha(s)$ is the equilibrium excess return on the risky asset, $\theta(x,t,s)$ is the portfolio allocation to the risky asset by investors with expertise level x at time s, $c(t,s)$ is consumption, F_{xx} is the maintenance cost, and T is the optimal exit date.

Proposition 4 ***Value and Policy Functions:*** *The value functions are given by*

$$\begin{aligned} V^x\left(w\left(t,s\right),x\right) &= y^x\left(x,t,s\right)\frac{w\left(t,s\right)^{1-\gamma}}{1-\gamma} && (17) \\ V^n\left(w\left(t,s\right),x\right) &= y^n\left(x,t,s\right)\frac{w\left(t,s\right)^{1-\gamma}}{1-\gamma} && (18) \end{aligned}$$

where $y^x(x)$ and $y^n(x)$ are given by:

$$\begin{aligned} y^x\left(x\right) &= \left[\frac{\left(\gamma-1\right)\left(r_f-f_{xx}\right)+\rho}{\gamma} + \frac{\left(\gamma-1\right)\alpha^2}{2\gamma^2\sigma^2\left(x\right)}\right]^{-\gamma} \quad \textit{and} && (19) \\ y^n\left(x\right) &= \left[\frac{\left(\gamma-1\right)r_f+\rho}{\gamma}\right]^{-\gamma}. && (20) \end{aligned}$$

The optimal policy functions $c^x(x,t,s)$, $c^n(t,s)$, and $\theta(x)$ are given by:

$$\begin{aligned} c^x\left(x,t,s\right) &= \left[y^x\left(x\right)\right]^{-\frac{1}{\gamma}} w\left(t,s\right), && (21) \\ c^n\left(t,s\right) &= \left[y^n\left(x\right)\right]^{-\frac{1}{\gamma}} w\left(t,s\right) \quad \textit{and} && (22) \\ \theta\left(x,t,s\right) &= \frac{\alpha(t,s)}{\gamma\sigma^2\left(x\right)}. && (23) \end{aligned}$$

Furthermore, the wealth of experts evolves according to the law of motion:

$$\frac{dw\left(t,s\right)}{w\left(t,s\right)} = \left(\frac{r_f-f_{xx}-\rho}{\gamma} + \frac{\left(\gamma+1\right)\alpha^2\left(t,s\right)}{2\gamma^2\sigma^2\left(x\right)}\right) dt + \frac{\alpha\left(t,s\right)}{\gamma\sigma\left(x\right)} dB\left(t,s\right) \quad (24)$$

Finally, Investors choose to be experts if the excess return on the risky asset is high enough relative to the expertise-specific variance they face in order to compensate them for the maintenance *cost, given their common coefficient of relative risk aversion.*

$$\frac{\alpha^2(t,s)}{2\sigma^2(x)\gamma} \geq f_{xx}. \tag{25}$$

We prove this Proposition by guess and verify in the Appendix. Note that the law of motion for wealth is a sort of weighted average of the return to the risky and riskless assets, as determined by portfolio choice, net of consumption. Accordingly, as is standard and as can be easily shown, the drift and volatility of investors' wealth are increasing in the allocation to the risky asset. This will have important implications for the wealth distribution in the stationary equilibrium of our model.

4.2 The Distribution of Expert Wealth

The total amount of wealth allocated to the complex risky asset, as well as the distribution of expert wealth across expertise levels, are key aggregate state variables for the the first and second moments of the equilibrium returns to the complex risky asset. Once the entry decision has been made, given that we do not clear the market for the riskless asset, the wealth of non-experts is irrelevant for the returns to the complex risky asset. We solve for the cross-sectional distribution of expert wealth in a stationary equilibrium of our model. Given that expertise is fixed over time for each investor, constructing the wealth distribution at each expertise level is sufficient to obtain the cross-sectional joint distribution of wealth and expertise.

First, we note that in order to construct a stationary equilibrium given that experts' wealth on average grows over time, it is convenient to study the ratio of individual wealth to the mean wealth of agents with highest expertise, $E[w^{\bar{x}}(t)]$.

$$z(t,s) = \frac{w(t,s)}{E[w^{\bar{x}}(t)]}.$$

Then, the ratio $z(t,s)$ follows a geometric Brownian motion given by

$$\frac{dz(t,s)}{z(t,s)} = \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2(t,s)}{2\gamma^2\sigma^2(x)} - g(\bar{x})\right)dt + \frac{\alpha(t,s)}{\gamma\sigma(x)}dB(t,s), \tag{26}$$

where $\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2(t)}{2\gamma^2\sigma^2(x)} - g(\bar{x}) < 0$ represents the negative drift, or growth rate. Let the cross-sectional distribution of expert investors' wealth and expertise at time t be denoted by $\phi^x(z,x,t)$. Without additional assumptions, the relative wealth of lower expertise agents will shrink to zero. Two methods are then commonly used to ensure a stationary distribution. The first, for example used in Benhabib et al. [2014], is to employ a life cycle model, or Poisson elimination of agents. The second, employed by Gabaix [1999], is to introduce a reflecting barrier at a minimum wealth share, $z_{\min}$.[20]

We adopt the assumption of a minimum wealth share because it leads to a simpler wealth distribution. We will show that the stationary distribution of wealth at each expertise level will be a Pareto distribution. Adopting the assumption of Poisson death with a fixed initial wealth, for example, would instead lead to a double Pareto distribution, with a cutoff at the initial value of wealth. For example, see Benhabib et al. [2014] for the wealth distribution under the alternative assumption of Poisson elimination in a closely related model. This is also the assumption we adopt in our quantitative study in Eisfeldt et al. [2015].

We note that, since the reflecting boundary mainly affects low wealth investors, decisions near the boundary matter little for equilibrium pricing. However, we adopt an interpretation of exit and entry at $z_{\min}$ which ensures that policies are not distorted there. Then, since both time and state variables are continuous in our model, if policies are not distorted at $z_{\min}$, then they will not be distorted elsewhere. We discuss the interpretation we adopt in detail in the Appendix. The strategy we employ is to ensure that the value at $z_{\min}$ from adopting the optimal policy functions under non-reflecting wealth share dynamics is equal to the value of adopting those policies given that with some probability the agent will be punished by being forced to exit, and with some probability the agent will be rewarded by being able to infuse funds themselves, or by receiving new external funds. In the case of exit, we assume the investor is replaced by a new entrant with wealth share $z_{\min}$.

The law of motion for the mean wealth of the highest expertise agents is given by

$$dEw^{\bar{x}}(t) = [g(\bar{x})]\,dt.$$

where $g(\bar{x})$ will be determined in equilibrium. Note that the reflecting barrier at $z_{\min}$ implies that the growth rate of any individual agent, even those with the highest level of expertise, will grow more slowly than the mean wealth of the highest expertise agents.

We derive the Kolmogorov forward equations describing the evolution of the wealth distri-

[20]Gabaix [1999] constructs a model of the city size distribution, and thus his share variable represents relative population shares. See also the Appendix of that paper for a related method of constructing a stationary distribution using a Kesten [1973] process, which introduces a random shock with a positive mean to normalized city size.

bution, taking $\alpha(t)$ as given, as follows:[21]

$$\begin{aligned}
\partial_t \phi^x(z,x,t) = & -\partial_z \left(\left(z(r_f + \theta(x,t)\,\alpha(t)) - [y^x(x)]^{-\frac{1}{\gamma}} - f_{xx} - g(\bar{x}) \right) \phi^x(z,x,t) \right) \\
& + \frac{1}{2}\partial_{zz} \left([z\theta(x,t)\,\sigma(x)]^2 \phi^x(z,x,t) \right) \\
& = -\partial_z \left[\left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\,\alpha^2(t)}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right) \phi^x(z,x,t) \right] \\
& + \frac{1}{2}\partial_{zz} \left[\left(z\frac{\alpha(t)}{\gamma\sigma(x)} \right)^2 \phi^x(z,x,t) \right].
\end{aligned} \tag{27}$$

We then study the stationary distribution of wealth shares, in which $\partial_t \phi^x(z,x,t) = 0$. We take as given, for now, that $\alpha(t)$ will be constant as in the stationary equilibrium we define in the following section. This will be true given a stationary distribution over investors' individual state variables. A stationary distribution of wealth shares $\phi^x(z,x)$ satisfies the following equation

$$\begin{aligned}
0 \;=\; & -\partial_z \left[\left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\,\alpha^2}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right) \phi^x(z,x) \right] \\
& + \frac{1}{2}\partial_{zz} \left[\left(z\frac{\alpha}{\gamma\sigma(x)} \right)^2 \phi^x(z,x) \right].
\end{aligned} \tag{28}$$

We use guess and verify to show that the stationary distribution of wealth shares at each level of expertise is given by a Pareto distribution with an expertise specific tail parameter. This tail parameter, which we denote by β, is determined by the drift and volatility of the expertise specific law of motion for wealth shares. Intuitively, the larger the drift and volatility of the expertise specific wealth process, the fatter the tail of the wealth distribution at that level of expertise will be.

Proposition 5 *The stationary distribution of wealth shares $\phi^x(z,x)$ has the following form:*

$$\phi(z,x) \propto C(x) z^{-\beta(x)-1}, \tag{29}$$

[21] See Dixit and Pindyck [1994] for a heuristic derivation, or Karlin and Taylor [1981] for more detail.

where

$$\begin{aligned}\beta(x) &= C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma \geq 1, & (30)\\ C_1 &= 2\gamma\left(f_{xx} + \rho - r_f + \gamma g(\bar{x})\right), & (31)\\ C(x) &= \frac{1}{\int z^{-\beta} dz} = \frac{C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma}{z_{\min}^{-C_1 \frac{\sigma^2(x)}{\alpha^2} + \gamma}}. & (32)\end{aligned}$$

See the Appendix for the Proof, where we also show that, in the stationary distribution, $\beta > 1$, which ensures a finite integral, and confirms that the distribution satisfies stationarity. The following Corollary, which we also prove in the Appendix, gives the tail parameters for the highest expertise agents, as well as all other investors.

Corollary 6 *For the highest expertise agents, we have*

$$\beta(\bar{x}) = \frac{1}{1 - z_{\min}/\bar{z}} = C_1 \frac{\sigma^2(\bar{x})}{\alpha^2} - \gamma$$

where $\bar{z}$ is mean of normalized wealth of experts with highest expertise,

$$\bar{z} = \int_{z_{\min}}^{\infty} z\phi(z, \bar{x}) dz = z_{\min}\left[1 + \frac{1}{\beta(\bar{x}) - 1}\right]$$

and

$$g(\bar{x}) = \frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2}{2\gamma\sigma^2(\bar{x})} + \frac{\alpha^2}{2\gamma^2\sigma^2(\bar{x})} \frac{1}{1 - z_{\min}/\bar{z}}$$

For all other expertise levels, we have

$$\beta(x) = \left(\gamma + \frac{z_{\min}/\bar{z}}{1 - z_{\min}/\bar{z}}\right) \frac{\sigma^2(x)}{\sigma^2(\bar{x})} - \gamma > 1$$

The parameter β controls the tail of each expertise specific wealth distribution. The smaller is β, the more slowly the distribution decays, and the fatter is the upper tail. Clearly, β is an increasing function of risk aversion, γ, and an increasing function of expertise level volatility, $\sigma(x)$. Since expertise level effective volatility is decreasing in x, the wealth distribution of experts with a higher level of fixed expertise has a fatter tail. This makes intuitive sense because investors with higher expertise tend to allocate more wealth to the risky asset, which increases the mean and volatility of their wealth growth rate. Both a higher drift, and a wider distribution of shocks, lead to a fatter upper tail for wealth. However, perhaps surprisingly, as Lemma 7 illustrates, not every input which increases difference in the fraction of wealth allocated to the risky asset leads to an increase in the degree of fat tails of the expertise specific wealth distributions. See the Appendix for the proof.

Lemma 7 ***Relation Between*** $\theta(x)$ ***and*** $\beta(x)$

Consider two level of expertise, $x_{\min}$ *and* $x_{\max}$*, we have*

$$\theta(x_{\max}) - \theta(x_{\min}) = \frac{\alpha}{\gamma} \frac{\sigma^2(x_{\min}) - \sigma^2(x_{\max})}{\sigma^2(x_{\max})\,\sigma^2(x_{\min})},$$

and

$$\beta(x_{\max}) - \beta(x_{\min}) = 2\gamma^2 \left(f_{xx} + r - r_f + \gamma g(\bar{x})\right) \frac{\sigma^2(x_{\max})\,\sigma^2(x_{\min})}{\alpha^3} \left[\theta(x_{\min}) - \theta(x_{\max})\right].$$

If a larger difference portfolio choice is due to either a higher excess return or a lower risk aversion, the dispersion in β *is smaller. If it is due to an increase in the difference in effective volatilities, then the difference in* β*'s is larger.*

We can also measure the degree of wealth inequality within each expertise level as $\frac{1}{\beta(x)}$. High expertise levels exhibit greater size "inequality". Corollary 8 then shows that an increase between the difference in the allocation to the risky asset by high and low expertise investors increases the difference in size differences between small and large funds across expertise levels.

Corollary 8 *We have*

$$\frac{1}{\beta(x_{\max}) + \gamma} - \frac{1}{\beta(x_{\min}) + \gamma} = \frac{\alpha\left[\theta(x_{\min}) - \theta(x_{\max})\right]}{2\left(f_{xx} + r - r_f + \gamma g(\bar{x})\right)}$$

Therefore, a higher dispersion in portfolio choice leads to a higher dispersion in inequality across expertise levels.

4.3 Aggregation and Stationary Equilibrium

In this section, we define a stationary equilibrium, and state the condition which determines the market clearing α in a stationary equilibrium. We also perform comparative statics over aggregate demand for the complex risky asset under general and specific assumptions for $\sigma(x)$

and $\lambda(x)$. Slightly abusing notation, we define aggregate investment in the complex risky asset to be I, given each sum of expertise level investment $I(x)$ $\forall x$, where:

$$I = \int \lambda(x) I(x) \, dx. \tag{33}$$

Because I, and hence α, are functions of the stationary distribution for wealth, and in turn the stationary distribution for wealth is a function of α, it is difficult to show comparative statics over α directly in closed form. We proceed in two ways to study comparative statics for α, individual, and market level Sharpe ratios. First, we study the general case. For comparative statics over the equilibrium excess return α, we use the fact that, in the stationary equilibrium we study, α is a bijection of I, and perform comparative statics over I. Second, we compute numerical comparative statics under specific parametric assumptions. With investors' policy functions and the joint distribution of wealth and expertise in closed form, numerical solutions for the fixed point α are straightforward.

We first define a stationary equilibrium. In order to ensure that the supply of the complex risky asset does not become negligible as investor wealth grows, we assume that the supply grows proportionally to the mean wealth of the highest expertise investors. That is, we assume:

$$S(t) = Sg(\bar{x})\, t.$$

For convenience, we assume that the support of expertise is bounded above by $\bar{x}$, although most of our results only require that $\sigma(x)$ satisfies $\lim_{x\to\infty} \sigma(x) = \underline{\sigma} > 0$.

Definition 9 *A stationary equilibrium consists of a market clearing α, policy functions for all investors, and a stationary distribution over investor types $i \in \{x, n\}$, expertise levels x, and wealth shares z, $\phi(i, z, x, t)$, such that given an initial wealth distribution, an expertise distribution $\lambda(x)$, and parameters $\{\gamma, \rho, S, r_f, f_{nx}, f_{xx}, \sigma_v\}$ the economy satisfies:*

1. *Investor optimality: Investors choose the optimal stopping time to enter into, or to exit from, the complex risky asset market according to Equation (25), as well as optimal consumption and portfolio choices $\{c^n(t), c(x,t), \theta(x,t)\}_{t=0}^{\infty}$ according to Equations (21)-(23), such that their utilities are maximized.*

2. *Market clearing: The equilibrium market clearing α is determined by equating supply and demand:*

$$S(t) = \int \lambda(x,t)\, \theta(x,t) \left(W(x,t) - c(x,t)\right) dx.$$

In a stationary equilibrium, we have:

$$I \equiv \int \lambda(x) I(x) \, dx = S,$$

Define $Z(x)$ to be the total expertise level wealth share,

$$Z(x) = z_{\min}\left(1 + \frac{1}{\beta(x) - 1}\right).$$

Then, define $I(x)$ to be the detrended total expertise level investment in the complex risky asset, namely,

$$\begin{aligned} I(x) &= \frac{\alpha}{\gamma\sigma^2(x)}\left(1 - [y^x(x)]^{-\frac{1}{\gamma}}\right) Z(x), \\ &= \frac{\alpha}{\gamma\sigma^2(x)}\left[\frac{-(\gamma - 1)(r_f - f_{xx}) + \gamma - \rho}{\gamma} - \frac{(\gamma - 1)\alpha^2}{2\gamma^2\sigma^2(x)}\right] Z(x). \end{aligned}$$

3. *The distribution over all individual state variables is stationary, i.e. $\partial_t \phi(i, z, x, t) = 0$.*

Aggregate Demand and α We cannot express the equilibrium excess return in closed form. However, the following Proposition shows that the equilibrium excess return, α, and aggregate demand, I, form a bijection. This uniqueness result in turn ensures that α can be numerically solved for as the unique fixed point to the above equation.

We first describe the comparative statics for demand conditional on investors' expertise levels in Lemma 10.

Lemma 10 *Using the market clearing condition, we have following comparative statics. As long as:*

$$y^x(\bar{x}) < \frac{1}{2},$$

we have that,

1. $\frac{\partial I(x)}{\partial d} > 0$, *where* $d = \frac{\alpha^2}{\sigma^2(x)}$
2. $\frac{\partial I(x)}{\partial \alpha} > 0$
3. $\frac{\partial I(x)}{\partial \sigma_v} < 0$
4. $\frac{\partial I(x)}{\partial \gamma} < 0$

If we have that $y^x(\bar{x}) < \frac{1}{1+\beta(\beta-1)}$*, then we also have:*

5. $\frac{\partial I(x)}{\partial f_{xx}} < 0$

With expertise level total demands in hand, we can construct the aggregate result.

Proposition 11 *Aggregate market demand for the complex risky asset is an increasing function of the excess return,* α*, and* α *and* I *form a bijection. Mathematically,*

$$\frac{\partial I}{\partial \alpha} > 0,$$

as long as

$$y^x(\bar{x}) < \frac{1}{3},$$

that is, if the consumption of all experts is less than a third of their total wealth.

The condition in Proposition 11 is stronger than what is needed. The Appendix gives some weaker conditions, along with a proof. The reason that a restriction on consumption is sufficient is because such a condition rules out the case in which wealth effects from improved investment opportunities are too strong.

Proposition 12 provides comparative statics over the aggregate demand for the complex risky asset, I. Using the result in Proposition 11, these comparative statics also hold for α. The proof, as well as weaker conditions for the results (but with longer expressions), appear in the Appendix. We have that the demand for the risky asset at each level of expertise is increasing in the squared investor specific Sharpe ratio. Thus, as long as each investor specific Sharpe ratio increases, aggregate demand will increase.[22] In addition, demand for the risky asset is decreasing in fundamental volatility, risk aversion, and the maintenance cost. As a result, α is increasing in fundamental volatility, risk aversion, and the maintenance cost. If, as seems intuitive, an increase in these parameters proxies for greater asset complexity, then α will be higher in more complex asset markets.

Proposition 12 *Using the market clearing condition, we have following comparative statics hold: As long as:*

$$y^x(\bar{x}) < \frac{1}{2},$$

[22] In Proposition 14 below, we provide a condition under which all investor specific Sharpe ratios will move in the same direction. We also note that since higher expertise agents own a larger fraction of the risky asset, as long as they are unconstrained demand will tend to move in the same direction as their Sharpe ratios.

we have that:

1. $\frac{\partial I}{\partial \sigma_{\upsilon}} < 0$, *thus* α *is an increasing function of fundamental risk*

2. $\frac{\partial I}{\partial \gamma} < 0$, *thus* α *is an increasing function of risk aversion*

   If we have that $y^{x}(\bar{x}) < \frac{1}{1+\beta(\beta-1)}$, *then we also have:*

3. $\frac{\partial I}{\partial f_{xx}} < 0$, *thus* α *is an increasing function of the maintenance cost.*

We now turn to showing that the equilibrium required excess return on the complex risky asset is decreasing in the amount of wealth commanded by agents with higher levels of expertise. The wealth distribution at each expertise level is a Pareto distribution with an expertise specific tail parameter. Thus, by shifting the distribution of expertise rightward, leading to a new distribution with a relatively larger fraction of higher expertise investors, relatively more wealth will reside with agents with higher expertise. Thus, with any rightward shift, the joint distribution of wealth and expertise becomes more efficient. There is also a direct effect on overall wealth from a rightward shift in the distribution of expertise, since the wealth distribution at higher expertise levels exhibits fatter right tails. Accordingly, Proposition 13 shows that if the density of experts shifts to the right, then demand for the complex risky asset will increase, and the required equilibrium excess return will decrease. Thus, the equilibrium excess return is decreasing in the amount of wealth which resides in the hands of agents with higher expertise. The proof appears in the Appendix.

Proposition 13 *If* $\frac{\partial \sigma(x)}{\partial x} < 0$, *and* Λ_1 *exhibits first-order stochastic dominance over* Λ_2, $I(\Lambda_1) \geq I(\Lambda_2)$. *As a result* $\alpha(\Lambda_1) < \alpha(\Lambda_2)$.

Investor Specific and Market Level Sharpe ratios With the analysis of equilibrium excess returns in hand, we now turn to the equilibrium risk-return tradeoff at the investor and market level as described by the investor-specific, and market level Sharpe ratios. As in the static model, at the market level we define both the equally weighted and value weighted Sharpe ratios, but focus on the equally weighted Sharpe ratio in our analysis. In addition to offering cleaner comparative statics because it does not depend on investor portfolio choices and market shares, the equally weighted Sharpe ratio represents the expected Sharpe ratio that an investor who could pay a cost to draw from the expertise distribution above the entry cutoff would earn. In that sense, it is the "expected Sharpe ratio". Note that the Sharpe ratio for non-experts is not defined.

Given the excess return on the risky asset, we define the investor-specific Sharpe Ratio as:

$$SR(x) = \frac{\alpha}{\sigma(x)}.$$

We provide results for investor-specific Sharpe ratios under the three possible cases for the elasticity of investor specific risk with respect to fundamental volatility in Proposition 14.

Proposition 14 *The comparative statics for the investor-specific Sharpe ratios respect to fundamental volatility depend on which of the three possible cases for the elasticity of investor specific risk with respect to fundamental volatility is under study:*

1. *Case 1: If* $\frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ *is a constant, that is*

$$\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} = 0,$$

we must have that $SR(z,x)$ *is either an increasing or a decreasing function of fundamental risk for all expertise levels.*

2. *Case 2: If* $\frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ *is a decreasing function of expertise, that is*

$$\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} < 0,$$

then there is a cutoff level x^**, such that for all* $x < x^*$*, we have* $\frac{\partial SR(x)}{\partial \sigma_v} < 0$*; and for all* $x > x^*$*, we have* $\frac{\partial SR(x)}{\partial \sigma_v} > 0$.

3. *Case 3: If* $\frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ *is an increasing function of expertise, that is*

$$\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} > 0,$$

there is a cutoff level x^**, such that for all* $x < x^*$*, we have* $\frac{\partial SR(x)}{\partial \sigma_v} > 0$*; and for all* $x > x^*$*, we have* $\frac{\partial SR(x)}{\partial \sigma_v} < 0$*. Further,* x^* *exists if for any small* $\varepsilon < 10^{-6}$

$$(0, \varepsilon) \subseteq \left\{ \frac{\partial \log \sigma(x)}{\partial \log \sigma_v} | \text{ for all } x \right\} \subseteq [0, \infty).$$

The following are examples of the three cases for the elasticity of investor specific risk with respect to fundamental volatility in Proposition 14. Note that the static model uses a functional form that satisfies case 2, as shown in Lemma 3 part 6.

Case 1: $\frac{\partial \log \sigma(x)}{\partial \log \sigma_\upsilon}$ is a constant, $\sigma(x) = \left(a + x^{-b}\right) \sigma_\upsilon^2$

Case 2: $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\upsilon}}{\partial x} < 0$, $\sigma(x) = a + x^{-b} \sigma_\upsilon^2$ [23]

Case 3: $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\upsilon}}{\partial x} > 0$, $\sigma(x) = a\sigma_\upsilon^2 + x^{-b}$

We define the equally weighted market equilibrium Sharpe ratio as:

$$SR^{ew} = E\left[\frac{\alpha}{\sigma(x)} | \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right].$$

We show in the Appendix that the value weighted market equilibrium Sharpe ratio is given by:

$$\begin{aligned} SR^{vw} &= E\left[\frac{\theta(z-c)}{I}\frac{\alpha}{\sigma(x)} | \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right] \\ &= \frac{\alpha}{\gamma I} E\left[\frac{1 - [y^x(x)]^{-\frac{1}{\gamma}}}{\sigma^3(x)} Z(x) | \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right]. \end{aligned}$$

Further Results with Specific Parametric Assumptions We further make two additional assumptions about the distribution of expertise and the functional form for the expertise specific variance in order to expand our intuition of how the equilibrium excess return and Sharpe ratio depends on the parameters of joint distribution of wealth and expertise. Specifically, we assume that:

$$\sigma^2(x) = \left(a + x^{-b}\right) \sigma_\upsilon^2$$

and that x is log-normally distributed

$$\lambda(x) \sim \log N\left(\mu_x - \frac{\sigma_x^2}{2}, \sigma_x^2\right).$$

The mean level expertise value is μ_x, which is invariant to the volatility of expertise, σ_x. Note that the assumption for $\sigma^2(x)$ satisfies case 1 of Proposition 14.

Proposition 15 gives results for comparative statics over aggregate demand under our given parametric assumptions. The result of Proposition 11 holds for these assumptions, and thus

[23] x^{-b} can be replaced by any function $f(x)$ as long as $\frac{\partial f(x)}{\partial x} < 0$.

any increase in demand will lead to an increase in the equilibrium excess return. The proof appears in the Appendix.

Proposition 15 *Using the market clearing condition, we have following comparative statics*

1. $\frac{\partial I}{\partial a} > 0$
2. $\frac{\partial I}{\partial \mu_x} > 0$ *if* $\frac{1}{2a} \log\left[\frac{2\gamma f_{xx}}{\alpha^2}\right] \geq \exp\left(\mu_x - \frac{1}{2\sigma_x^2}\right)$
3. $\frac{\partial I}{\partial \sigma_x} > 0$ *if* $\frac{1}{2a} \log\left[\frac{2\gamma f_{xx}}{\alpha^2}\right] \geq \exp\left(\mu_x - \frac{1}{2\sigma_x^2} + \sigma_x\right)$
4. $\frac{\partial I}{\partial \sigma_\upsilon} < 0$

Demand increases with the mean and variance of expertise. The reason demand increases with the variance of expertise, is because it is the right tail that mainly matters for demand. Note that demand decreases with respect to the fundamental variance. The reason is that the endogenous decrease in the allocation to the risky asset outweighs the direct effect of the increase in fundamental volatility on the fatness of tails of the expertise-specific the wealth distributions. Note also that we must have this result in the parametric example, since Proposition 12 shows that it holds in the general case.

To help decompose the effects on the equally weighted market equilibrium Sharpe ratio, Lemma 16 begins by providing comparative statics on the equally weighted market variance in equilibrium. The proof appears in the appendix.

Lemma 16 *Define the equally weighted market variance to be*

$$E\left[\sigma^2(x)\,|\sigma^2(x) \leq \frac{\alpha^2}{2\gamma f_{xx}}\right],$$

then, we have following results in partial equilibrium:

1. $\frac{\partial \mathbb{E}\left[\sigma^2(x)\right]}{\partial f_{xx}} < 0$
2. $\frac{\partial \mathbb{E}\left[\sigma^2(x)\right]}{\partial \gamma} < 0$
3. $\frac{\partial \mathbb{E}\left[\sigma^2(x)\right]}{\partial a} > 0$
4. $\frac{\partial \mathbb{E}\left[\sigma^2(x)\right]}{\partial \sigma_\upsilon} > 0$

Next, proposition 17 provides comparative statics on the equally weighted market equilibrium Sharpe ratio.

Proposition 17 *Using the market clearing condition, we have following comparative statics for the equally weighted market Sharpe ratio in partial equilibrium:*

$$SR^{ew} = E\left[\frac{\alpha}{\sigma(x)} | \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right].$$

1. $\frac{\partial SR^{ew}}{\partial a} < 0$

2. $\frac{\partial SR^{ew}}{\partial \sigma_\upsilon} < 0$

3. $\frac{\partial SR^{ew}}{\partial \alpha} > 0$

Proof. Direct Calculation. ■

4.4 Numerical Example

This section presents complementary numerical results and comparative statics. The model generates closed form policy functions and wealth distributions conditional on expertise levels. We solve the model in both partial equilibrium and general equilibrium. In partial equilibrium, the excess return is given exogenously, and held fixed, while aggregate demand (and implicitly supply) varies. In general equilibrium, the excess return is computed endogenously given a fixed supply of the risky asset. Because α and I form a bijection (Proposition 11 provides conditions for which they are one to one and onto), for any given supply of the complex risky asset, we can solve for the market equilibrium α in the following steps:

1. Choose an upper and a lower bound for α, namely α_1 and α_2, ($\alpha_1 > \alpha_2$).

2. Let $\alpha = \frac{\alpha_1 + \alpha_2}{2}$, and compute the total demand for the risky asset

$$\int \lambda(x) I(x) dx$$

3. If $S - \int \lambda(x) I(x) dx < -10^{-4}$, let $\alpha_1 = \alpha$ and back to step 1; if $S - \int \lambda(x) I(x) dx > 10^{-4}$, let $\alpha_2 = \alpha$ and back to step 1; otherwise, STOP.

Our baseline parameters are summarized in Table 1. The time interval is one quarter. The risk-free rate is 1%. The discount factor is 1%. The maintenance cost is also 1%. The coefficient of relative risk-aversion is 5. The log-normal distribution of expertise has a mean of 0 and volatility of 5. The minimum wealth share is set to 0.4. The fundamental standard deviation of the risky asset return is 20%. We choose a functional form for expertise-specific variance with $a = 0.28$ and $b = 2$. This implies that the highest expertise investors can eliminate 47% of fundamental risk, and face a standard deviation of 10.5%. The model generates a stationary equilibrium. The entry cutoff is $x = 1.45$, which implies that the total measure of experts is 47%. The average wealth of experts is 0.33. In aggregate, experts invest 88.2% of their total wealth in risky assets.

Next, we perform some comparative statics over the parameters of the model. Figure 1 plots the model statistics with different values for the excess return in partial equilibrium. Both the standard deviation of investor specific variance and the market Sharpe ratio are equally weighted in our computations. The demand for the risky asset is an increasing function of the excess return. With a higher α, the entry cutoff is lower and therefore there are more experts in the market. The market equally weighted standard deviation of the risky asset returns are an increasing function of α because we at a higher α there are more investors with low expertise in the market. However, the market Sharpe ratio is dominated by the increasing value of α, and thus the Sharpe ratio increases when α increases. All of these results are consistent with our analytical comparative statics.

Figures 2 - 6 show the model comparative statics in both general equilibrium and partial equilibrium. All blue lines represent model results in partial equilibrium with a fixed excess return and a perfectly elastic supply of the risky asset. All red lines represent model results in general equilibrium with a fixed supply of the risky asset and the market clearing equilibrium value for the excess return. Figure 2 plots model statistics with different values for the maintenance cost. A higher maintenance cost represents a higher cost of being an expert. There are fewer experts in both partial equilibrium and general equilibrium at higher maintenance costs. Demand for the risky asset is smaller in partial equilibrium as a result, resulting in a higher excess return in general equilibrium to clear the market. Also, the market equally weighted Sharpe ratio increases as the maintenance cost becomes larger. The higher maintenance cost represents an entry barrier for non-experts, and we argue that this cost proxies for asset complexity. More complex assets require larger investments in expertise. Experts earn a higher excess return with more complex strategies in which the higher investment costs deter entrants.

Figure 3 displays the model statistics as a function of risk aversion. In partial equilibrium, if investors are more risk averse, they invest less in the risk asset. Also, there are fewer experts in the market. Market risk is lower because of the selection effect of only "better" experts operating in the market. These results change somewhat in general equilibrium, since α increases in general equilibrium. The positive effects on entry because of the increase in α dominate the negative effects from increased γ. Thus, there are more experts in the market, and the worse selection of experts implies a higher market risk. However, the market Sharpe ratio increases in both general equilibrium and partial equilibrium as a result of increased risk aversion.

Figure 4 plots the model statistics with different fundamental risk levels. It shares some similar patterns with the effects of risk aversion. The results differ, however, for the market level risk and Sharpe ratio in partial equilibrium. With increasing fundamental risk, there are fewer experts in partial equilibrium and more experts in general equilibrium, parallel to the results for an increase in γ. However, the market risk is higher in both partial and general equilibrium because the increased value of fundamental risk dominates the selection effects on entry.

Figure 5 and Figure 6 show the model results with different value of b. b has two effects in our model. First, b represents the difference in investor-specific risk between high and low expertise investors. A higher b means a larger difference. Second, b controls the entry cutoff for experts. Because a higher b increases the effect of expertise at all expertise levels, a higher b results in a lower entry cutoff. Thus, this parameter has somewhat opposing effects. To see this, Figure 5 considers both effects, while Figure 6 eliminates the second effect. In Figure 5, with a higher value of b, we have a lower entry cutoff in partial equilibrium. The effects of lowered risk because of higher b dominate the negative selection effects on market risk. The standard deviation of market risk decreases as the value of b increases in both partial and general equilibrium. The decreased risk and increased fraction of experts results in a lower market excess return. α decreases faster than market risk in general equilibrium. Thus we see a higher Sharpe ratio in general equilibrium. In Figure 6, we counterbalance the selection effects of b on entry with by appropriately scaling the value of a to ensure that the entry cutoff does not change with varies value of b in partial equilibrium. In this way, we can focus on the comparative statics from variation in the complexity of the asset strategy, as proxied for by b. With a higher value of b, a has to be higher to keep the entry cutoff constant. Figure 6 displays several different implications for comparative statics over b. First, the standard deviation of overall market risk is an increasing function of b. More complex strategies, in which there is a bigger difference in the risk faced by high expertise investors vs. low expertise investors, display more market risk. Second, the market demand for the risky asset is a decreasing function of b because of this higher market risk. And, the general equilibrium excess return is an increasing function of b to compensate the higher risk. Third, the increased variance dominates the increased market return, so that the market Sharpe Ratio is a decreasing function of b in both partial and general equilibrium.

5 Conclusion

To be completed.

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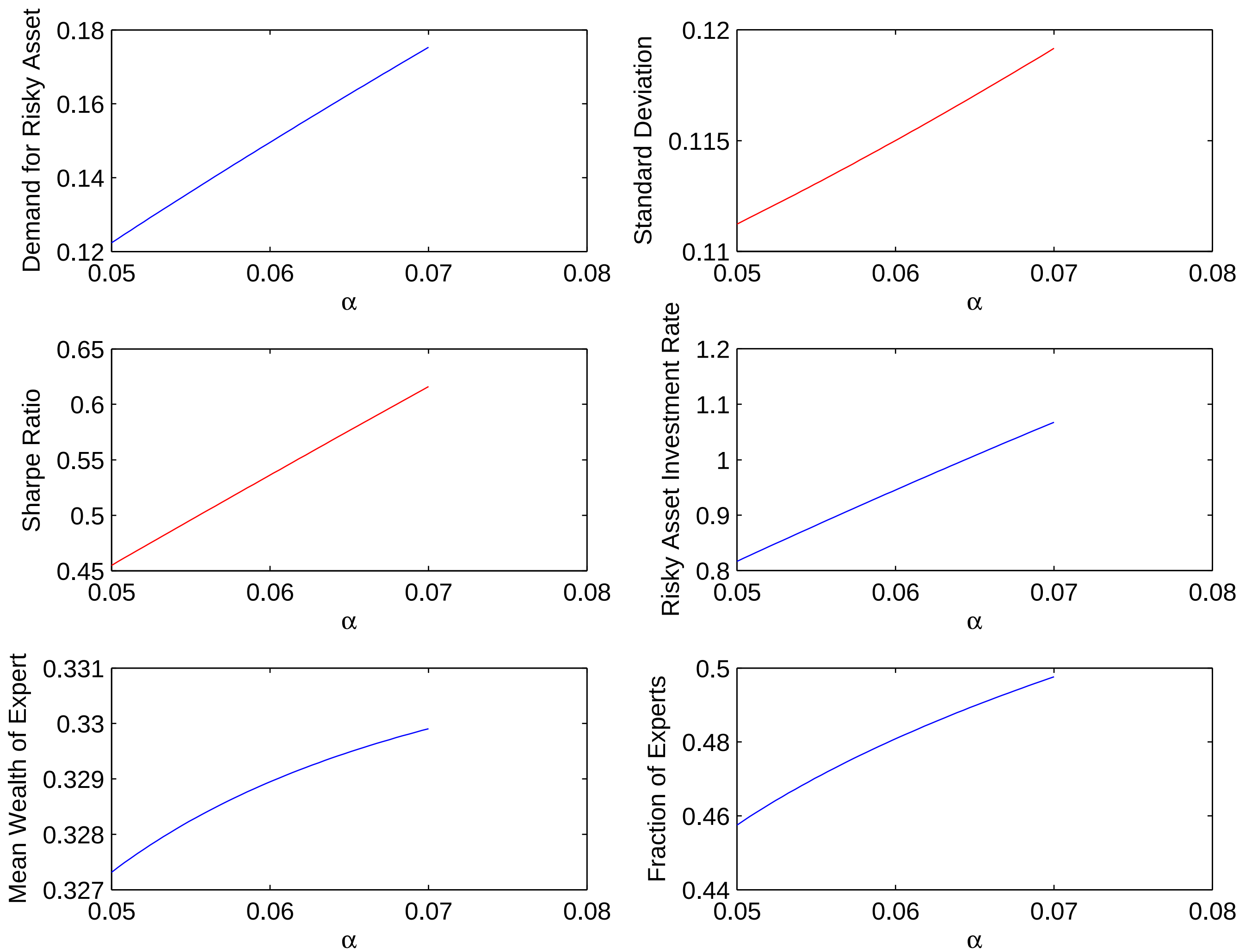

Figure 1: Model comparative statics: Excess Return

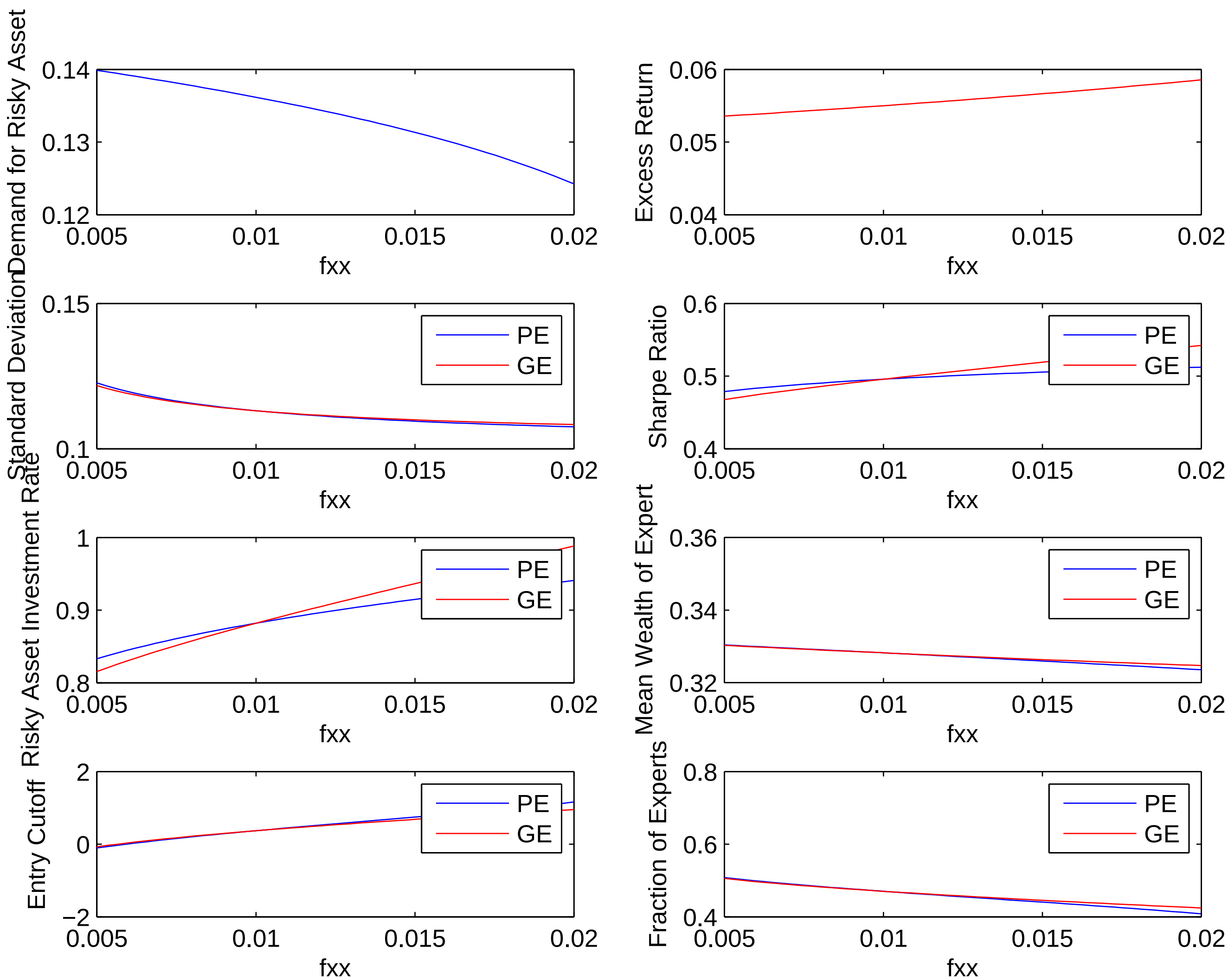

Figure 2: Model comparative statics: maintenance cost

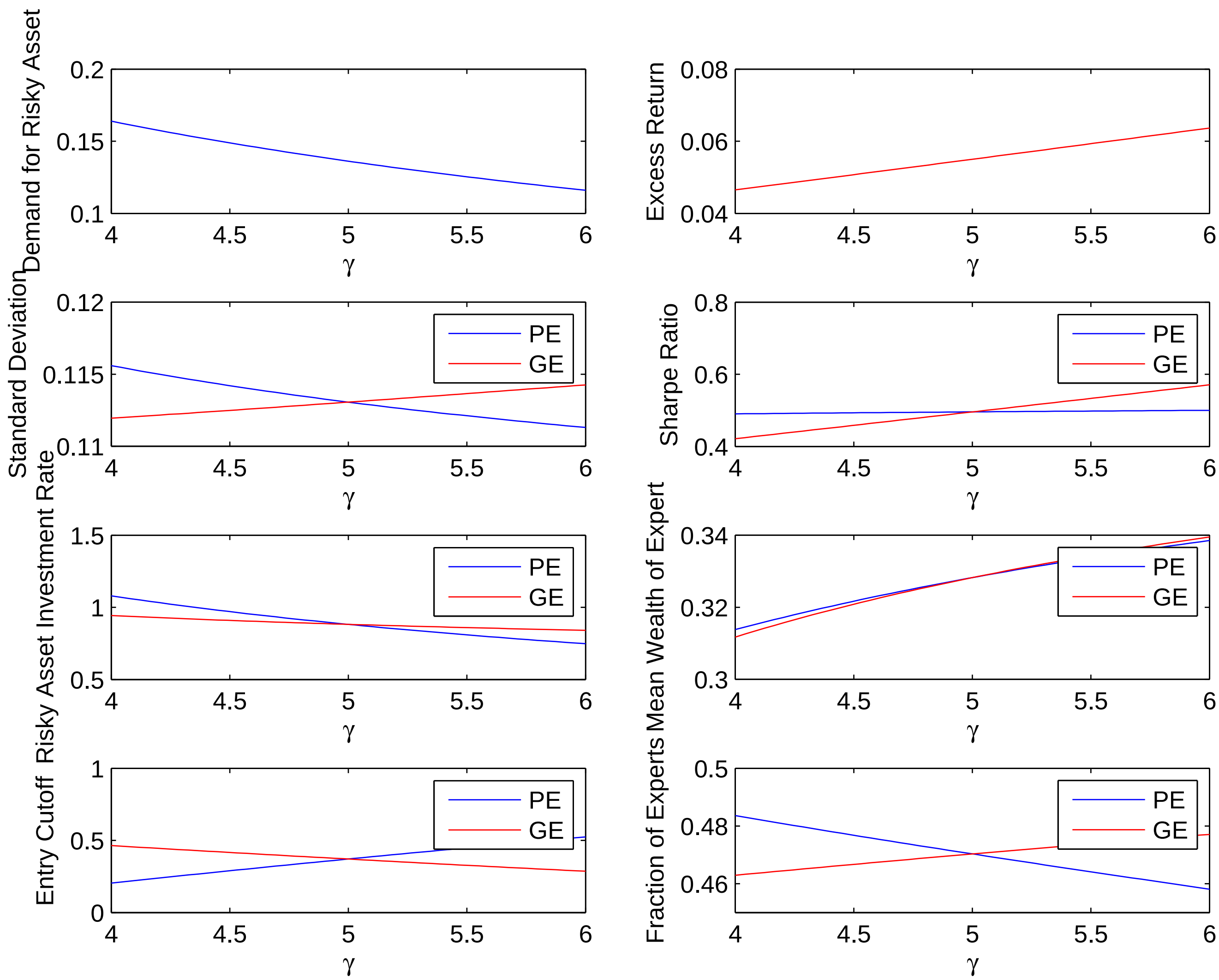

Figure 3: Model comparative statics: risk aversion

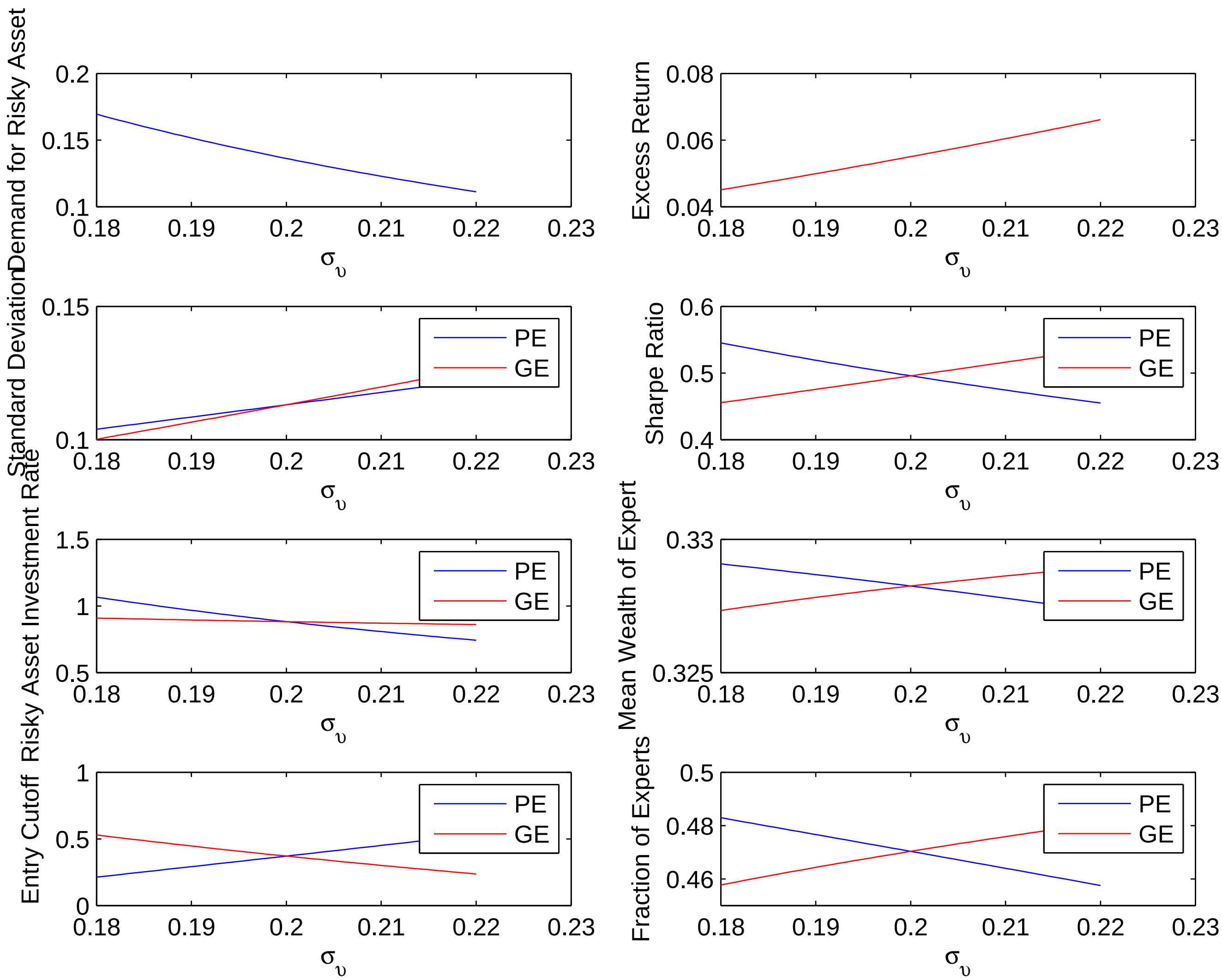

Figure 4: Model comparative statics: fundamental risk

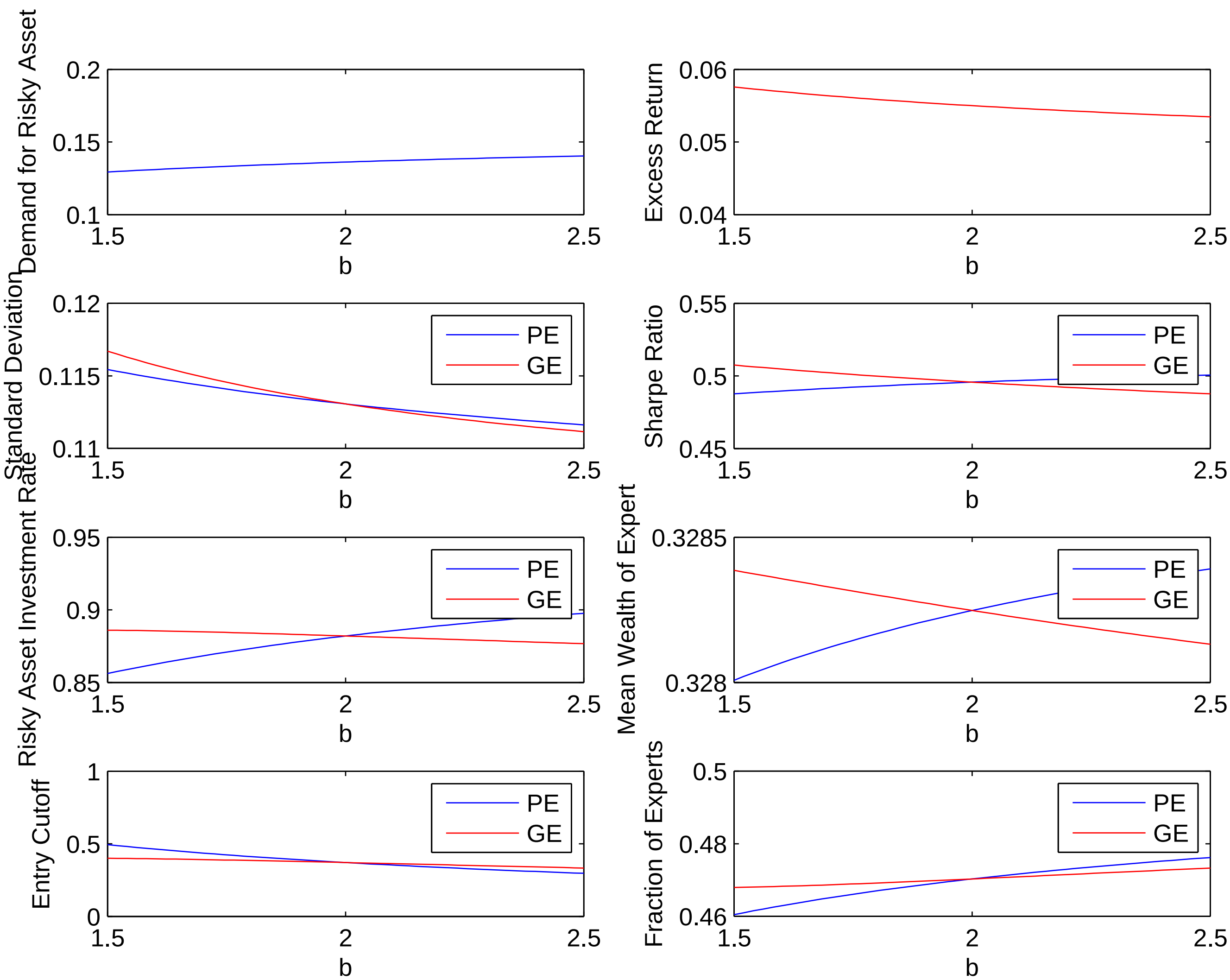

Figure 5: Model comparative statics: b with fixed a

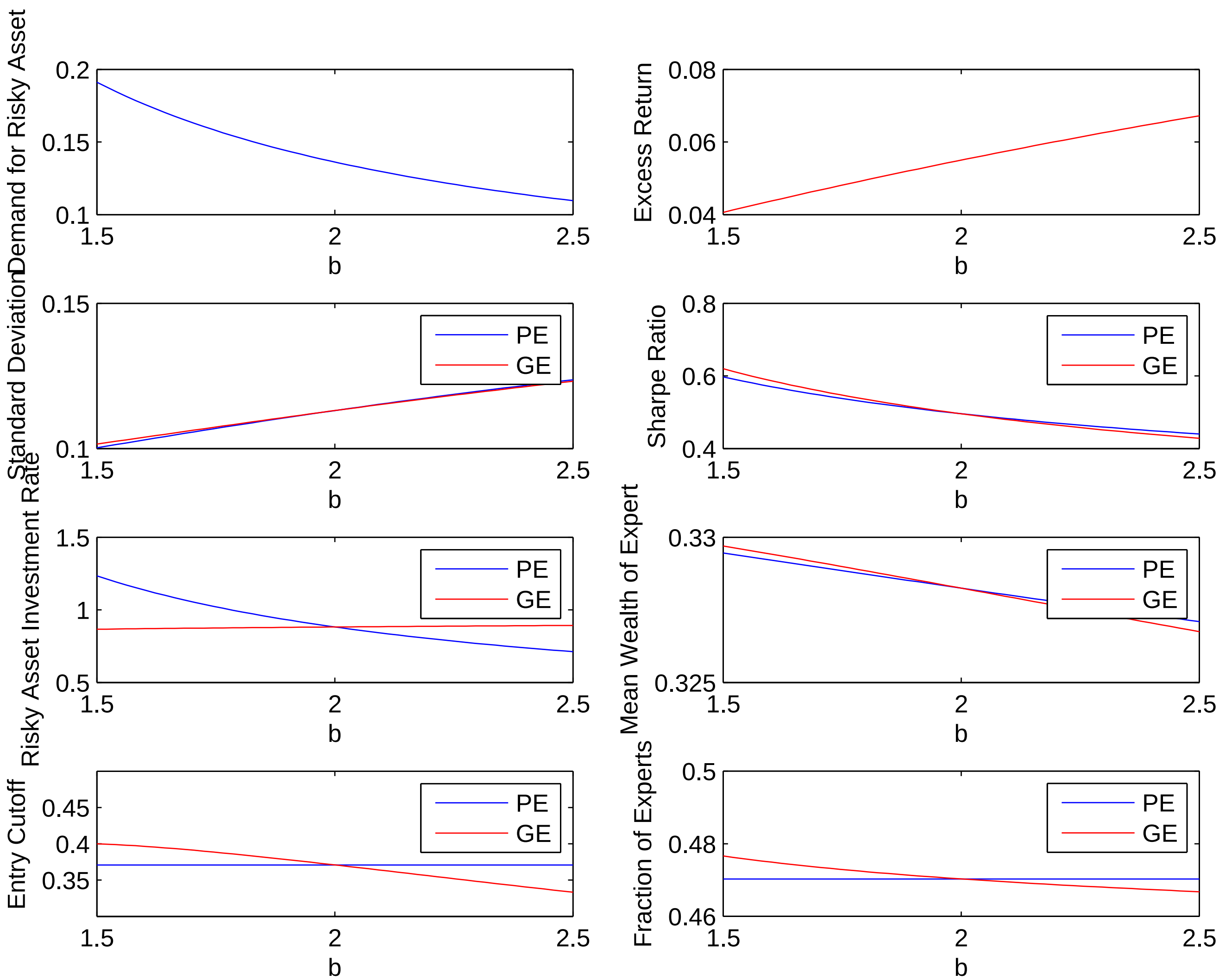

Figure 6: Model comparative statics: b with flexible a

Table 1: Parameter Values: Numerical Example for Market Clearing in the Dynamic Model:

Parameter	Symbol	Value	Target
Discount factor	ρ	0.01	Annual interest rate
Risk-free rate	r_f	0.01	Annual interest rate
Risk aversion	γ	5	Data/mean portfolio choice
Entry cost	fnx	0.03	
Maintenance cost	fxx	0.01	
Risky asset supply	S	0.14	$\alpha = 5.5\%$
Volatility of risky asset return	σ_v	20%	
Mean of expertise process	μ_x	0	
Volatility of expertise process	σ_x	5	
Constant in σ_x^2	a	0.28	
Slope of σ_x^2	b	2	
Minimum wealth	s_{min}	0.4	

Appendix A: Static Model

This section contains proofs and additional results for the static model.

Optimal Portfolio Choice

This section describes how to solve the optimal portfolio allocation problem in the static model. We also use upper case letters for level variables, and lower case letters for log variables. Under the assumptions in the main text, the optimization problem for an investor with wealth W and expertise X, can be written as:

$$v(W,X) = \max_{\theta} \mathbb{E}\left[\frac{(WR_p)^{1-\gamma}}{1-\gamma}\right]$$

subject to

$$\begin{aligned} R_p &= \theta R + (1-\theta) R_f, \\ r|(W,X) &\sim N\left(\boldsymbol{\mu} - \frac{1}{2}\frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X}\right). \end{aligned}$$

Campbell and Viceira [2002a] and Campbell and Viceira [2002b] show that the log portfolio return r_p over a short time horizon with bounded variance, can be approximated by:

$$r_p \approx r_f + \theta(r - r_f) + \frac{1}{2}\theta(1-\theta)\frac{\sigma_v^2}{X}.$$

As a result,

$$r_p|(W,X) \sim N\left(r_f + \theta(\boldsymbol{\mu} - r_f) - \frac{1}{2}\theta^2\frac{\sigma_v^2}{X}, \theta^2\frac{\sigma_v^2}{X}\right).$$

Then the value function equals:

$$v(W,X) = \max_{\theta} \frac{W^{1-\gamma}}{1-\gamma} \exp\left((1-\gamma) r_f + (1-\gamma)\theta(\boldsymbol{\mu} - r_f) - \frac{1}{2}\gamma(1-\gamma)\theta^2\frac{\sigma_v^2}{X}\right).$$

Hence, The investor's optimization problem becomes:

$$\max_{\theta} \left\{\theta(\boldsymbol{\mu} - r_f) - \frac{\gamma}{2}\theta^2\frac{\sigma_v^2}{X}\right\}.$$

Equilibrium Market Excess Return

This section describes how to derive the equilibrium market excess return, $\boldsymbol{\alpha}$, from the log expected return, $\boldsymbol{\mu}$, given all parameters. Because

$$r|\left(W,X\right)\sim N\left(\boldsymbol{\mu}-\frac{1}{2}\frac{\sigma_v^2}{X},\frac{\sigma_v^2}{X}\right),$$

Then

$$\mathbb{E}\left(R|W,X\right)=\exp\left(\boldsymbol{\mu}\right).$$

In addition,

$$\mathbb{E}\left[R\right]=\mathbb{E}\left[\mathbb{E}\left(R|W,X\right)\right].$$

Hence,

$$\mathbb{E}\left[R\right]=\exp\left(\boldsymbol{\mu}\right).$$

Finally,

$$\alpha=\exp\left(\boldsymbol{\mu}\right)-R_f.$$

Equilibrium Equally Weighted Market Sharpe ratio

This section describes how to derive the equilibrium equally weighted market Sharpe ratio, SR, from the log expected return, $\boldsymbol{\mu}$, given all parameters. Because

$$r|\left(W,X\right)\sim N\left(\boldsymbol{\mu}-\frac{1}{2}\frac{\sigma_v^2}{X},\frac{\sigma_v^2}{X}\right),$$

Then

$$\boldsymbol{Var}\left(R|W,X\right)=\exp\left(2\boldsymbol{\mu}\right)\left(\exp\left(\frac{\sigma_v^2}{X}\right)-1\right).$$

In addition, we have proven that

$$\mathbb{E}\left(R|W,X\right)=\mathbb{E}\left[R\right]=\exp\left(\boldsymbol{\mu}\right).$$

Therefore, the equally weighted variance of the risky asset, is given by:

$$\boldsymbol{Var}\left[R\right]=\mathbb{E}\left(\boldsymbol{Var}\left(R|W,X\right)\right)=\exp\left(2\boldsymbol{\mu}\right)\left(\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X}\right)\right]-1\right).$$

Hence, the equally weighted market Sharpe ratio, can be written as:

$$SR=\frac{1-R_f\exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X}\right)\right]-1}},$$

where $\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X}\right)\right]=\sum_{k=0}^{\infty}\frac{1}{k!}\sigma_v^{2k}\mathbb{E}[\exp(-kx)]$, using a Taylor expansion of $\exp(\sigma_v^2X^{-1})=1+\sigma_v^2X^{-1}+\frac{1}{2!}\sigma_v^4X^{-2}+\frac{1}{3!}\sigma_v^6X^{-3}+\ldots$, which is equivalent to:

$$\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X}\right)\right]=\sum_{k=0}^{\infty}\frac{1}{k!}\sigma_v^{2k}[\exp(-k\mu_x+\frac{1}{2}k^2\sigma_x^2)],$$

where we have used the moment-generating function of the normal distribution. Hence, the equally weighted market Sharpe ratio, can be written as:

$$SR=\frac{1-R_f\exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\sum_{k=0}^{\infty}\frac{1}{k!}\sigma_v^{2k}\exp(-k\mu_x+\frac{1}{2}k^2\sigma_x^2)-1}},$$

Provided that σ_v^4 is small enough, this SR is approximately equal to the following expression:

$$SR\approx\frac{1-R_f\exp\left(-\boldsymbol{\mu}\right)}{\sigma_v\exp(-\frac{1}{2}\mu_x+\frac{1}{4}\sigma_x^2)}.$$

Equilibrium Investor-Specific Sharpe ratio

This section describes how to derive the equilibrium investor-specific Sharpe ratio, $SR(W, X)$, from log expected return, $\boldsymbol{\mu}$, given all parameters. For an investor with wealth W and expertise X, Because

$$r|\left(W, X\right) \sim N\left(\boldsymbol{\mu} - \frac{1}{2}\frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X}\right),$$

Then

$$\mathbb{E}\left(R|W, X\right) = \exp\left(\boldsymbol{\mu}\right),$$

And

$$\boldsymbol{Var}\left(R|W, X\right) = \exp\left(2\boldsymbol{\mu}\right)\left(\exp\left(\frac{\sigma_v^2}{X}\right) - 1\right).$$

Hence, the investor-specific Sharpe ratio is given by:

$$SR\left(W, X\right) = \frac{1 - R_f \exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}$$

$$E\left[SR\left(W, X\right)\right] = E\left[\frac{1 - R_f \exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}\right]$$

Proof of Lemma 2 and 3

This section describes how to prove lemma 2 and 3. From equations (23), (7), (8) and (9), we can derive that, if η denotes any parameter $\eta \in \{\gamma, S, \mu_w, \sigma_w, \rho_{w,x}\}$:

1. $\frac{\partial \boldsymbol{\alpha}}{\partial \eta} = \exp(\boldsymbol{\mu}) \frac{\partial \boldsymbol{\mu}}{\partial \eta}$;
2. $\frac{\partial (SR)}{\partial \eta} = \frac{R_f \exp(-\boldsymbol{\mu})}{\sqrt{\mathbb{E}\left(\exp\left(\frac{\sigma_\upsilon^2}{X}\right)\right)-1}} \frac{\partial \boldsymbol{\mu}}{\partial \eta}$;
3. $\frac{\partial SR(W,X)}{\partial \eta} = \frac{R_f \exp(-\boldsymbol{\mu})}{\sqrt{\exp\left(\frac{\sigma_\upsilon^2}{X}\right)-1}} \frac{\partial \boldsymbol{\mu}}{\partial \eta}$;
4. $\frac{\partial \boldsymbol{Var}(SR(W,X))}{\partial \eta} = 2\left(1 - R_f \exp(-\boldsymbol{\mu})\right) R_f \exp(-\boldsymbol{\mu}) \boldsymbol{Var}\left(\frac{1}{\sqrt{\exp\left(\frac{\sigma_\upsilon^2}{X}\right)-1}}\right) \frac{\partial \boldsymbol{\mu}}{\partial \eta}$;
5. $\frac{\partial^2 SR(W,X)}{\partial \eta \partial X} = \frac{\partial\left(\frac{R_f \exp(-\boldsymbol{\mu})}{\sqrt{\exp\left(\frac{\sigma_\upsilon^2}{X}\right)-1}}\right)}{\partial\left(\frac{\sigma_\upsilon^2}{X}\right)} \frac{\partial\left(\frac{\sigma_\upsilon^2}{X}\right)}{\partial X} \frac{\partial \boldsymbol{\mu}}{\partial \eta}$;
6. $\frac{\partial^2 \theta^*(W,X)}{\partial \eta \partial X} = \frac{1}{\gamma \sigma_\upsilon^2} \frac{\partial \boldsymbol{\mu}}{\partial \eta}, \forall \eta \neq \gamma$, and $\frac{\partial \theta^*(W,X)}{\partial \gamma} = 0$
7. $\frac{\partial SR(W,X)}{\partial \sigma_\upsilon^2} = \frac{R_f \exp(-\boldsymbol{\mu}) \frac{\boldsymbol{\mu} - r_f}{\sigma_\upsilon^2} - \frac{1}{2}\left(1 - R_f \exp(-\boldsymbol{\mu})\right) \frac{\exp\left(\frac{\sigma_\upsilon^2}{X}\right)\frac{1}{X}}{\exp\left(\frac{\sigma_\upsilon^2}{X}\right)-1}}{\sqrt{\exp\left(\frac{\sigma_\upsilon^2}{X}\right)-1}}$.

Hence, $Sign\left(\frac{\partial \boldsymbol{\mu}}{\partial \eta}\right) = Sign\left(\frac{\partial \boldsymbol{\alpha}}{\partial \eta}\right) = Sign\left(\frac{\partial (SR)}{\partial \eta}\right) = Sign\left(\frac{\partial SR(W,X)}{\partial \eta}\right)$
$= Sign\left(\frac{\partial \boldsymbol{Var}(SR(W,X))}{\partial \eta}\right) = Sign\left(\frac{\partial^2 SR(W,X)}{\partial \eta \partial X}\right) = Sign\left(\frac{\partial^2 \theta^*(W,X)}{\partial \eta \partial X}\right)$

In addition, we have:

1. Because $\frac{\exp\left(\frac{\sigma_v^2}{X}\right)\frac{1}{X}}{\exp\left(\frac{\sigma_v^2}{X}\right)-1} > \frac{1}{X}$, $\forall X$,

   then $\frac{\partial SR(W,X)}{\partial \sigma_v^2} < \frac{R_f \exp(-\boldsymbol{\mu})\frac{\boldsymbol{\mu}-r_f}{\sigma_v^2} - \frac{1}{2}\left(1-R_f \exp(-\boldsymbol{\mu})\right)\frac{1}{X}}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right)-1}} < 0$, $\forall X < \underline{X}$,

   where $\underline{X} = \frac{\frac{1}{2}\left(1-R_f \exp(-\boldsymbol{\mu})\right)}{R_f \exp(-\boldsymbol{\mu})\frac{\boldsymbol{\mu}-r_f}{\sigma_v^2}} > 0$;

2. $0 = \frac{1}{\sigma_v^2}\left(1+\frac{\sigma_v^2}{X}\right) - \frac{1}{X} - \frac{1}{\sigma_v^2} < \frac{1}{\sigma_v^2}\exp\left(\frac{\sigma_v^2}{X}\right) - \frac{1}{X} - \frac{1}{\sigma_v^2} = \left(\exp\left(\frac{\sigma_v^2}{X}\right)-1\right)\left(\frac{1}{X}+\frac{1}{\sigma_v^2}\right) - \exp\left(\frac{\sigma_v^2}{X}\right)\frac{1}{X}$,

   then $\frac{\exp\left(\frac{\sigma_v^2}{X}\right)\frac{1}{X}}{\exp\left(\frac{\sigma_v^2}{X}\right)-1} < \frac{1}{X} + \frac{1}{\sigma_v^2}$,

   and $\frac{\partial SR(W,X)}{\partial \sigma_v^2} > \frac{R_f \exp(-\boldsymbol{\mu})\frac{\boldsymbol{\mu}-r_f}{\sigma_v^2} - \frac{1}{2}\left(1-R_f \exp(-\boldsymbol{\mu})\right)\left(\frac{1}{X}+\frac{1}{\sigma_v^2}\right)}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right)-1}}$, $\forall X$.

   Hence, if $\bar{X} = \frac{1}{\frac{R_f \exp(-\boldsymbol{\mu})\frac{\boldsymbol{\mu}-r_f}{\sigma_v^2}}{\frac{1}{2}\left(1-R_f \exp(-\boldsymbol{\mu})\right)} - \frac{1}{\sigma_v^2}} > 0$, then $\forall X > \bar{X}, \frac{\partial SR(W,X)}{\partial \sigma_v^2} > 0$.

   $\bar{X} > 0$ if and only if $F(\boldsymbol{\mu}-r_f) \equiv \exp\left(-\left(\boldsymbol{\mu}-r_f\right)\right)\left(\boldsymbol{\mu}-r_f+\frac{1}{2}\right) - \frac{1}{2} > 0$.
   We can prove $F(\boldsymbol{\mu}-r_f) > 0$ if and only if $0 < \boldsymbol{\mu}-r_f < 1.2564$, but $\boldsymbol{\mu}-r_f = 1.2564$ corresponds to an $\boldsymbol{\alpha}$ around 250%. Then we can conclude $\bar{X} > 0$ for all reasonable parameters.

3. we can prove by direct computation that $\bar{X} > \underline{X}$ whenever $\bar{X} > 0$.

In sum, for all reasonable parameters, $\exists \bar{X} > \underline{X} > 0$ such that $\forall X > \bar{X}, \frac{\partial SR(W,X)}{\partial \sigma_v^2} > 0$, and $\forall X < \underline{X}, \frac{\partial SR(W,X)}{\partial \sigma_v^2} < 0$. The general functional form for effective risk yields similar results.

Equilibrium Value-Weighted Market Sharpe ratio

This section shows that our main conclusions still hold with respect to the value-weighted equilibrium market Sharpe ratio. Because

$$r|\left(W,X\right) \sim N\left(\boldsymbol{\mu} - \frac{1}{2}\frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X}\right),$$

Then

$$\mathbb{E}\left(R|W,X\right) = \exp\left(\boldsymbol{\mu}\right).$$

Then the value-weighted market expected return also equals to:

$$\exp\left(\boldsymbol{\mu}\right),$$

In addition,

$$\boldsymbol{Var}\left(R|W,X\right) = \exp\left(2\boldsymbol{\mu}\right)\left(\exp\left(\frac{\sigma_v^2}{X}\right) - 1\right).$$

Therefore, the value-weighted variance of the risky asset, is given by:

$$\int\int \boldsymbol{Var}\left(R|W,X\right)\frac{\exp(w)\theta^*\left(\exp(x)\right)f(w,x)}{\int\int \exp(w)\theta^*\left(\exp(x)\right)f(w,x)\,dw\,de}\,dw\,de,$$

Which equals to

$$\exp\left(2\boldsymbol{\mu}\right)\frac{\mathbb{E}\left[\left(\exp\left(\frac{\sigma_v^2}{X}\right) - 1\right)\exp(w+x)\right]}{\boldsymbol{X}}$$

Hence, the value-weighted market Sharpe ratio, can be written as:

$$\frac{1 - R_f e^{-\mu}}{\sqrt{\frac{\mathbb{E}\left[\left(\exp\left(\frac{\sigma_v^2}{X}\right) - 1\right)\exp(w+x)\right]}{\boldsymbol{X}}}}.$$

where $\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X} + w + x\right)\right]$, using a Taylor expansion of $\exp(\sigma_v^2 X^{-1} + w + x) = 1 + \sigma_v^2 X^{-1} + w + x + \frac{1}{2!}(\sigma_v^2 X^{-1} + w + x)^2 + \frac{1}{3!}(\sigma_v^2 X^{-1} + w + x)^3 + \ldots$, which is equivalent to:

$$\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X} + w + x\right)\right] = \sum_{k=0}^{\infty}\frac{1}{k!}(\sigma_v^2 X^{-1} + w + x)^k)],$$

This will be approximately equal to

$$\begin{aligned} E[\exp(\sigma_v^2 X^{-1} + w + x)] \quad &\approx 1 + E[\sigma_v^2 X^{-1}] + E[w] + E[x], \\ &+ \frac{1}{2}E[\sigma_v^4 X^{-2} + w^2 + x^2 + 2wx\sigma_v^4 X^{-2} + 2w^2 x\sigma_v^2 X^{-1} + 2wx^2\sigma_v^2 X^{-1}] \end{aligned}$$

The moment-generating function is given by:

$$M(t_1, t_2) = E[\exp(t_1 w)\exp(t_2 x)] = \exp\left(t_1\mu_x + t_2\mu_w + (1/2)(t_1\sigma_x^2 + t_2\sigma_w^2 + 2t_1 t_2 wx\rho_{w,x}\sigma_x\sigma_w)\right)$$

$$\partial M(t_1, t_2)$$

Then, if η denotes any parameter $\eta \in \{\gamma, S\}$,

$$\frac{\delta\,(SR)}{\delta\eta} = \frac{R_f e^{-\mu}}{\sqrt{\frac{\mathbb{E}\left[\left(\exp\left(\frac{\sigma_v^2}{X}\right)-1\right)\exp(w+x)\right]}{\boldsymbol{X}}}}\frac{\delta\mu}{\delta\eta},$$

However, unlike in the case of the equally weighted market equilibrium Sharpe ratio, for the value weighted Sharpe ratio, the derivatives needed to sign the comparative statics in lemma 2 and 3 for $\eta \in \{\mu_w, \sigma_w, \rho_{w,x}\}$ are indeterminate.

Wealth effect of Expertise

This section shows that while savings rates can theoretically be slightly decreasing in expertise, due to the wealth effect from higher expertise and the associated larger present value of investment opportunities, this effect tends to be dominated by the portfolio choice effect.

The static model with a consumption savings decision can be written as:

$$v(W,X)=\max_{(I,\theta)}\frac{(W-I)^{1-\gamma}}{1-\gamma}+\beta I^{1-\gamma}\mathbb{E}\left[\frac{R_p^{1-\gamma}}{1-\gamma}\right]$$

subject to:

$$\begin{aligned} R_p &= \theta R+(1-\theta)R_f, \\ r|\left(W,X\right) &\sim N\left(\boldsymbol{\mu}-\frac{1}{2}\frac{\sigma_v^2}{X},\frac{\sigma_v^2}{X}\right). \end{aligned}$$

Clearly, the portfolio choice problem is independent from the consumption savings decision, and the solution to the portfolio choice problem coincides with that of the static model without the consumption saving decision. For any choice of investment I, the optimal portfolio allocation always solves the same problem, maximizing the expected utility derived from the chosen investment level, given the return process for the riskless and risky assets. Therefore, we can plug the optimal portfolio choice back into the value function, and then derive the optimal investment. Finally we get:

$$I^*=W\frac{\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}$$

where

$$E\left[R_p^{1-\gamma}\right]=\exp\left(\left(1-\gamma\right)r_f+\frac{1}{2}\frac{\left(1-\gamma\right)}{\gamma}\frac{\left(\boldsymbol{\mu}-r_f\right)^2}{\frac{\sigma_v^2}{X}}\right).$$

Then, we can show that:

$$\frac{\partial I^*}{\partial X}=W\frac{\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}{\left(1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}\right)^2}\frac{1}{2}\frac{\left(\gamma-1\right)}{\gamma^2}\frac{\left(\boldsymbol{\mu}-r_f\right)^2}{\left(\frac{\sigma_v^2}{X}\right)^2}\frac{\partial\left(\frac{\sigma_v^2}{X}\right)}{\partial X}.$$

Observe that the saving rate decreases with the expertise if and only if $\gamma>1$.

However, for the investment in the risky asset, $I^*\theta^*$, we have:

$$I^*\theta^* = W\frac{\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}\frac{(\boldsymbol{\mu}-r_f)}{\gamma\frac{\sigma_v^2}{X}}.$$

Then,

$$\frac{\partial\left(I^*\theta^*\right)}{\partial X} = W\frac{\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}\frac{(\boldsymbol{\mu}-r_f)}{\gamma\left(\frac{\sigma_v^2}{X}\right)^2}\left(\frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{(\boldsymbol{\mu}-r_f)^2}{\frac{\sigma_v^2}{X}}\frac{1}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}-1\right)\frac{\partial\left(\frac{\sigma_v^2}{X}\right)}{\partial X}$$

There are two cases, depending on the coefficient of relative risk aversion:

1. If $\gamma < 1$, the saving rate does not fall with the expertise, neither does the investment in the risky asset.
   We have $\frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{(\boldsymbol{\mu}-r_f)^2}{\frac{\sigma_v^2}{X}}\frac{1}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}-1<0$.
   Therefore, $\frac{\partial(I^*\theta^*)}{\partial X}>0, \forall X$.
2. If $\gamma > 1$, the saving rate falls with the expertise, while the investment in the risky asset doesn't, as long as the expertise level is not too high.
   We have $\frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{(\boldsymbol{\mu}-r_f)^2}{\frac{\sigma_v^2}{X}}\frac{1}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}-1<\frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{(\boldsymbol{\mu}-r_f)^2}{\frac{\sigma_v^2}{X}}-1$, $\forall X$.
   Then $\frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{(\boldsymbol{\mu}-r_f)^2}{\frac{\sigma_v^2}{X}}\frac{1}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}-1<0$, $\forall X<\bar{X}$, where $\bar{X}=\frac{2\gamma^2}{(\gamma-1)}\frac{\sigma_v^2}{(\boldsymbol{\mu}-r_f)^2}$.
   Therefore, $\frac{\partial(I^*\theta^*)}{\partial X}>0, \forall X<\bar{X}$. The signs for comparative statics for $\forall X>\bar{X}$ are indeterminate.

In sum, investment in the risky asset increases with expertise, as long as the expertise level is not too high. The general functional form for effective risk yields similar results.

Appendix B: Dynamic Model

Proof. Proposition 4. We prove this Proposition by guess and verify. First, we write the HJB equations of our model

$$\begin{aligned}
\max_{c^x(t,s),\theta(x,t,s)} 0 &= u\left(c^x\left(t,s\right)\right) + V_w^x\left[w\left(t,s\right)\left(r_f + \theta\left(x,t,s\right)\alpha\left(t,s\right)\right) - c^x\left(t,s\right) - f_{xx}w\left(t,s\right)\right] \\
&\quad \frac{\theta^2\left(x\right)\sigma^2\left(x\right)w\left(t,s\right)^2}{2}V_{ww}^x - \rho V^x \\
\max_{c^n(t,s)} 0 &= u^n\left(c\left(t,s\right)\right) + V_w^n\left(r_f w\left(t,s\right) - c^n\left(t,s\right)\right) - \rho V^n
\end{aligned}$$

The first order conditions are

$$\begin{aligned}
u'\left(c\left(t,s\right)\right) &= V_w^x, \\
u'\left(c\left(t,s\right)\right) &= V_w^n, \\
V_w^x\alpha\left(t,s\right) + \theta\left(x,t,s\right)\sigma^2\left(x\right)w\left(t,s\right)V_{ww}^x &= 0.
\end{aligned}$$

Next, we guess that

$$\begin{aligned}
V^x\left(w\left(t,s\right),x\right) &= y^x\left(x,t,s\right)\frac{w\left(t,s\right)^{1-\gamma}}{1-\gamma}, \\
V^n\left(w\left(t,s\right),x\right) &= y^n\left(x,t,s\right)\frac{w\left(t,s\right)^{1-\gamma}}{1-\gamma}.
\end{aligned}$$

Thus

$$\begin{aligned}
c^x &= \left[y^x\left(x,t,s\right)\right]^{-\frac{1}{\gamma}}w\left(t,s\right), \\
c^n &= \left[y^n\left(x,t,s\right)\right]^{-\frac{1}{\gamma}}w\left(t,s\right),
\end{aligned}$$

and portfolio choice is given by

$$\theta\left(x,t,s\right) = \frac{\alpha\left(t,s\right)}{\gamma\sigma^2\left(x\right)}.$$

Plugging these choices into the HJB equations, we get

$$\begin{aligned}
0 &= \left[y^x\left(x,t,s\right)\right]^{-\frac{1-\gamma}{\gamma}} + y^x\left(x,t,s\right)\left(r_f + \frac{\alpha^2\left(t,s\right)}{\gamma\sigma^2\left(x\right)} - \left[y^x\left(x,t,s\right)\right]^{-\frac{1}{\gamma}} - f_{xx}\right)\left(1-\gamma\right) \\
&\quad -\frac{\alpha^2\left(t,s\right)}{2\gamma\sigma^2\left(x\right)}y^x\left(x,t,s\right)\left(1-\gamma\right) - \rho y^x\left(x,t,s\right) \\
&= \gamma\left[y^x\left(x,t,s\right)\right]^{-\frac{1-\gamma}{\gamma}} + y^x\left(x,t,s\right)\left(r_f + \frac{\alpha^2\left(t,s\right)}{2\gamma\sigma^2\left(x\right)} - f_{xx}\right)\left(1-\gamma\right) - \rho y^x\left(x,t,s\right), \\
0 &= \gamma\left[y^n\left(x,t,s\right)\right]^{-\frac{1-\gamma}{\gamma}} + y^n\left(x,t,s\right)\left(1-\gamma\right)r_f - \rho y^n\left(x,t,s\right).
\end{aligned}$$

Rearranging the equations, we solve for $y^x(x,t,s)$ and $y^n(x,t,s)$,

$$\begin{aligned} y^x(x,t,s) &= \left[\frac{(\gamma-1)(r_f - f_{xx}) + \rho}{\gamma} + \frac{(\gamma-1)\alpha^2(t,s)}{2\gamma^2\sigma^2(x)}\right]^{-\gamma}, \\ y^n(x,t,s) &= \left[\frac{(\gamma-1)r_f + \rho}{\gamma}\right]^{-\gamma}. \end{aligned}$$

Given all policy functions, we get the experts' wealth growth rates:

$$\frac{dw(t,s)}{w(t,s)} = \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2(t,s)}{2\gamma^2\sigma^2(x)}\right)dt + \frac{\alpha(t,s)}{\gamma\sigma(x)}dB(t,s)$$

Finally, given homogeneity of the value functions in wealth, the participation cutoff is constructed by direct comparison between $y^x(x,t,s)$ and $y^n(x,t,s)$. ■

Interpretation of $z_{\min}$ We assume that one of two things can happen to an investor at $z_{\min}$. With probability q, the investor is eliminated from the market, and replaced with a new agent with wealth share $z_{\min}$ and the same expertise as the exiting agent. Note that elimination in isolation would cause the incumbent agent to be conservative, to avoid $z_{\min}$. With probability $1-q$, the agent is rewarded by being able to infuse funds themselves, or by receiving new external funds, and the wealth share reflects. Note that this reward in isolation would cause the agent to risk shift, to take advantage of limited liability at $z_{\min}$. We require that $E[V^x(z,x)_{true}] = qE[V^x(z,x)_{die}] + (1-q)E[V^x(z,x)_{reflect}]$, conditional on the optimal policies under the true wealth share dynamics. Since the value under the true, non-reflecting, dynamics lies between the punishment value of dying and the reward value of reflection, we conjecture that there exists some probability that this is the case. For simplicity, we assume that $V^x(z,x)_{die} = 0$.

Sketch of proof We sketch our proof strategy for showing that the optimal policies in the model with reflecting barrier z_{min} are equivalent to those in the original model under our assumptions of a zero value at death, which is traded off with the positive value of reflection. The sketch assumes an optimal exit date. We will modify this to include the exit decision.

Model 1:

$$\begin{aligned} V^x(w(t,s),x) &= \max_{c^x(t,s),T,\theta(x,t,s)} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)}u(c^x(t,s))\,ds + e^{-\rho(T-t)}V^n(w(t,s),x)\right] \\ \text{s.t. } dw(t,s) &= [w(t,s)(r_f + \theta(x,t,s)\alpha(t,s)) - c^x(t,s) - F_{xx}]\,ds \\ &\quad + w(t,s)\theta(x,t,s)\sigma(x)\,dB(t,s), \end{aligned}$$

Model 2:

$$\begin{aligned}
V^{y}\left(w\left(t,s\right),y\right) &= \max_{c^{y}(t,s),T,\theta(y,t,s)} \max\left\{V^{x}\left(w\left(t,s\right),y\right), \mathbb{E}\left[\begin{array}{c}\int_{t}^{s'} e^{-\rho(s-t)}u\left(c^{y}\left(t,s\right)\right)ds \\ +\left(1-q\right)e^{-\rho(s'-t)}V^{y}\left(w_{\min},y\right)\end{array}\right]\right\} \\
\text{s.t. } dw\left(t,s\right) &= \left[w\left(t,s\right)\left(r_{f}+\theta\left(y,t,s\right)\alpha\left(t,s\right)\right)-c^{y}\left(t,s\right)-F_{yy}\right]ds \\
&\quad +w\left(t,s\right)\theta\left(y,t,s\right)\sigma\left(y\right)dB\left(t,s\right)
\end{aligned}$$

Assume $F_{xx} = F_{yy}$. They are both linear in wealth. By definition, we have

$$V^{y}\left(w\left(t,s\right),x\right) = \left(1-q\right)V^{y}\left(w_{\min},x\right), \text{ for } w\left(t,s\right) \leq w_{\min}.$$

Define

$$q\left(w\left(t,s\right),w_{\min}\right) = 1-\left[\frac{w\left(t,s\right)}{w_{\min}}\right]^{1-\gamma}, \text{ for } w\left(t,s\right) \leq w_{\min}.$$

Therefore, we have

$$V^{x}\left(w\left(t,s\right),x\right) = \left(1-q\right)V^{x}\left(w_{\min},x\right), \text{ for } w\left(t,s\right) \leq w_{\min}.$$

It suffices to show that

$$V^{y}\left(w\left(t,s\right),x\right) = V^{x}\left(w\left(t,s\right),x\right), \text{ for all } x \text{ and } w\left(t,s\right),$$

when agent's wealth hits $w_{\min}$ before he/she exits the market. That is

$$\begin{aligned}
V^{y}\left(w\left(t,s\right),y\right) &= \max_{c^{y}(t,s),\theta(y,t,s)} \mathbb{E}\left[\int_{t}^{s'} e^{-\rho(s-t)}u\left(c^{y}\left(t,s\right)\right)ds+\left(1-q\right)e^{-\rho(s'-t)}V^{y}\left(w_{\min},y\right)\right] \\
\text{s.t. } dw\left(t,s\right) &= \left[w\left(t,s\right)\left(r_{f}+\theta\left(y,t,s\right)\alpha\left(t,s\right)\right)-c^{y}\left(t,s\right)-F_{yy}\right]ds \\
&\quad +w\left(t,s\right)\theta\left(y,t,s\right)\sigma\left(y\right)dB\left(t,s\right)
\end{aligned}$$

First,

$$\begin{aligned}
V^{y}\left(w_{\min},x\right) &= \mathbb{E}\left[\int_{t}^{s'} e^{-\rho(s-t)}u\left(c^{y}\left(t,s\right)\right)ds+\left(1-q\right)e^{-\rho(s'-t)}V^{y}\left(w_{\min},x\right)\right] \\
&\geq \mathbb{E}\left[\int_{t}^{s'} e^{-\rho(s-t)}u\left(c^{x}\left(t,s\right)\right)ds+\left(1-q\right)e^{-\rho(s'-t)}V^{y}\left(w_{\min},x\right)\right],
\end{aligned}$$

that is,

$$
\begin{aligned}
& E\int_t^{s'} e^{-\rho(s-t)} u\left(c^x\left(t,s\right)\right) ds \\
\leq\; & E\int_t^{s'} e^{-\rho(s-t)} u\left(c^y\left(t,s\right)\right) ds \\
=\; & \frac{1}{1-\mathbb{E}\left[\left(1-q\right)e^{-\rho(s'-t)}\right]} V^y\left(w_{\min},x\right).
\end{aligned}
$$

Second,

$$
\begin{aligned}
V^x\left(w_{\min},x\right) &= \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)} u\left(c^x\left(t,s\right)\right) ds + e^{-\rho(s'-t)} V^x\left(w\left(t,s'\right),x\right)\right] \\
&= \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)} u\left(c^x\left(t,s\right)\right) ds + \left(1-q\right) e^{-\rho(s'-t)} V^x\left(w_{\min},x\right)\right] \\
&\geq \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)} u\left(c^y\left(t,s\right)\right) ds + \left(1-q\right) e^{-\rho(s'-t)} V^x\left(w_{\min},x\right)\right],
\end{aligned}
$$

that is,

$$
\begin{aligned}
& E\int_t^{s'} e^{-\rho(s-t)} u\left(c^y\left(t,s\right)\right) ds \\
\leq\; & E\int_t^{s'} e^{-\rho(s-t)} u\left(c^x\left(t,s\right)\right) ds \\
=\; & \frac{1}{1-\mathbb{E}\left[\left(1-q\right)e^{-\rho(s'-t)}\right]} V^x\left(w_{\min},x\right).
\end{aligned}
$$

Therefore, we must have

$$
E\int_t^{s'} e^{-\rho(s-t)} u\left(c^y\left(t,s\right)\right) ds = E\int_t^{s'} e^{-\rho(s-t)} u\left(c^x\left(t,s\right)\right) ds,
$$

and

$$
V^y\left(w_{\min},x\right) = V^x\left(w_{\min},x\right).
$$

Next,

$$\begin{aligned} V^y(w(t,s),x) &= \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)}u(c^y(t,s))\,ds + (1-q)\,e^{-\rho(s'-t)}V^y(w_{\min},x)\right] \\ &\geq \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)}u(c^x(t,s))\,ds + (1-q)\,e^{-\rho(s'-t)}V^x(w_{\min},x)\right] \\ &= \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)}u(c^x(t,s))\,ds + e^{-\rho(s'-t)}V^x(w(t,s'),x)\right] \\ &= V^x(w(t,s),x),\ \text{ for all } w(t,s) \end{aligned}$$

with equality iff $c^x(t,s) = c^y(t,s)$ and $\theta^x(x,t,s) = \theta^y(x,t,s)$.

Lastly,

$$\begin{aligned} V^x(w(t,s),x) &= \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)}u(c^x(t,s))\,ds + e^{-\rho(s'-t)}V^x(w(t,s'),x)\right] \\ &\geq \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)}u(c^y(t,s))\,ds + (1-q)\,e^{-\rho(s'-t)}V^y(w_{\min},x)\right] \\ &= V^y(w(t,s),x),\ \text{ for all } w(t,s) \end{aligned}$$

with equality iff $c^x(t,s) = c^y(t,s)$ and $\theta^x(x,t,s) = \theta^y(x,t,s)$.

Therefore,

$$\begin{aligned} V^y(w(t,s),x) &= V^x(w(t,s),x),\ \text{ for all } x \text{ and } w(t,s). \\ c^x(t,s) &= c^y(t,s), \\ \theta^x(x,t,s) &= \theta^y(x,t,s). \end{aligned}$$

Proof. Proposition 5 We prove this Proposition by guess and verify. We guess that:

$$\phi(z,x) = Cz^{-\beta-1},$$

Then, we have

$$
\begin{aligned}
0 = & -\partial_z \left(z^{-\beta} \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\,\alpha^2}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right) \right) \\
& + \frac{1}{2}\partial_{zz} \left(z^{1-\beta} \frac{\alpha^2}{\gamma^2\sigma^2(x)} \right) \\
= & \ \beta \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\,\alpha^2}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right) \\
& - \frac{1}{2}\beta(1-\beta) \left[\frac{\alpha}{\gamma\sigma(x)} \right]^2 \\
= & \ \beta \left[\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2(\gamma+\beta)}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right]
\end{aligned}
$$

Thus

$$
\begin{aligned}
\beta &= C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma \geq 1, \\
C_1 &= 2\gamma\left(f_{xx} + \rho - r_f + \gamma g(\bar{x})\right), \\
C &= \frac{1}{\int z^{-\beta-1} dz} = \frac{C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma}{z_{\min}^{-C_1 \frac{\sigma^2(x)}{\alpha^2} + \gamma}}.
\end{aligned}
$$

Note there are two roots of equation

$$
0 = \beta \left[\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2(\gamma+\beta)}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right].
$$

We only take the root that is larger than 1 to ensure the mean wealth has a finite mean. ■

Proof. Corollary 6. For the highest expertise agents, we have

$$
\bar{z} = \int_{z_{\min}}^{\infty} z\phi(z,\bar{x})dz = \int_{z_{\min}}^{\infty} Cz^{-\beta(\bar{x})}dz = z_{\min}\left[1 + \frac{1}{\beta(\bar{x}) - 1}\right].
$$

This gives us another expression of $\beta(\bar{x})$,

$$
\beta(\bar{x}) = \frac{1}{1 - z_{\min}/\bar{z}}.
$$

Also, we know

$$
\beta(\bar{x}) = 2\gamma\left(f_{xx} + \rho - r_f + \gamma g(\bar{x})\right)\frac{\sigma^2(\bar{x})}{\alpha^2} - \gamma
$$

Therefore, we have

$$2\gamma\left(f_{xx}+\rho-r_f+\gamma g\left(\bar{x}\right)\right)\frac{\sigma^2\left(\bar{x}\right)}{\alpha^2}-\gamma=\frac{1}{1-z_{\min}/\bar{z}},$$

Rearrange the above equation, we get

$$g\left(\bar{x}\right)=\frac{r_f-f_{xx}-\rho}{\gamma}+\frac{\alpha^2}{2\gamma\sigma^2\left(\bar{x}\right)}+\frac{\alpha^2}{2\gamma^2\sigma^2\left(\bar{x}\right)}\frac{1}{1-z_{\min}/\bar{z}}.$$

Plug $g\left(\bar{x}\right)$ into $\beta\left(x\right)$, we derive

$$\beta\left(x\right)=\left(\gamma+\frac{z_{\min}/\bar{z}}{1-z_{\min}/\bar{z}}\right)\frac{\sigma^2\left(x\right)}{\sigma^2\left(\bar{x}\right)}-\gamma.$$

■

Proof. Lemma 7

$$\begin{aligned}\text{Recall that:}\theta\left(x\right) &= \frac{\alpha}{\gamma\sigma^2\left(x\right)}\\ \beta\left(x\right) &= 2\gamma\left(f_{xx}+r-r_f+\gamma g\left(\bar{x}\right)\right)\frac{\sigma^2\left(e\right)}{\alpha^2}-\gamma\end{aligned}$$

Consider two levels of expertise, $x_{\min}$ and $x_{\max}$, we have

$$\begin{aligned}\theta\left(x_{\max}\right)-\theta\left(x_{\min}\right) &= \frac{\alpha}{\gamma}\left[\frac{1}{\sigma^2\left(x_{\max}\right)}-\frac{1}{\sigma^2\left(x_{\min}\right)}\right]\\ &= \frac{\alpha}{\gamma}\frac{\sigma^2\left(x_{\min}\right)-\sigma^2\left(x_{\max}\right)}{\sigma^2\left(x_{\max}\right)\sigma^2\left(x_{\min}\right)},\end{aligned}$$

and

$$\begin{aligned}\beta\left(x_{\max}\right)-\beta\left(x_{\min}\right) &= 2\gamma\left(f_{xx}+r-r_f+\gamma g\left(\bar{x}\right)\right)\frac{1}{\alpha^2}\left[\sigma^2\left(x_{\max}\right)-\sigma^2\left(x_{\min}\right)\right]\\ &= 2\gamma^2\left(f_{xx}+r-r_f+\gamma g\left(\bar{x}\right)\right)\frac{\sigma^2\left(x_{\max}\right)\sigma^2\left(x_{\min}\right)}{\alpha^3}\left[\theta\left(x_{\min}\right)-\theta\left(x_{\max}\right)\right].\end{aligned}$$

If a larger dispersion of portfolio choice is due to either a higher excess return or a lower risk aversion, the dispersion in β is smaller, since:

$$\begin{aligned}\frac{\partial\left[\beta\left(x_{\max}\right)-\beta\left(x_{\min}\right)\right]}{\partial\alpha} &< 0,\text{ and }\frac{\partial\left[\theta\left(x_{\min}\right)-\theta\left(x_{\max}\right)\right]}{\partial\alpha}>0\\ \frac{\partial\left[\beta\left(x_{\max}\right)-\beta\left(x_{\min}\right)\right]}{\partial\gamma} &> 0,\text{ and }\frac{\partial\left[\theta\left(x_{\min}\right)-\theta\left(x_{\max}\right)\right]}{\partial\gamma}<0\end{aligned}$$

Consider the case where $\sigma^2(e_{\max})\sigma^2(e_{\min})$ is a constant, then

$$\frac{\partial\left[\beta\left(e_{\max}\right)-\beta\left(e_{\min}\right)\right]}{\partial\left[\theta\left(e_{\min}\right)-\theta\left(e_{\max}\right)\right]}=2\gamma^{2}\left(f_{xx}+r-r_{f}+\gamma g\left(\bar{e}\right)\right)\frac{\sigma^{2}\left(e_{\max}\right)\sigma^{2}\left(e_{\min}\right)}{\alpha^{3}}.$$

A larger dispersion in portfolio choice, resulting from a larger difference between effective volatility, implies a larger dispersion of tail distribution. The condition on the product of the effective variances is not necessary, however, as can be seen by simple algebra. ■

Proof. Lemma 8 Given the Pareto distribution, we can measure inequality as $\frac{1}{\beta(x)}$. We have

$$\begin{aligned}\frac{1}{\beta\left(x_{\max}\right)+\gamma}-\frac{1}{\beta\left(x_{\min}\right)+\gamma} &= \frac{1}{2\gamma\left(f_{xx}+r-r_{f}+\gamma g\left(\bar{x}\right)\right)}\alpha^{2}\left[\frac{1}{\sigma^{2}\left(x_{\max}\right)}-\frac{1}{\sigma^{2}\left(x_{\min}\right)}\right]\\ &= \frac{1}{2\left(f_{xx}+r-r_{f}+\gamma g\left(\bar{x}\right)\right)}\alpha\left[\theta\left(x_{\min}\right)-\theta\left(x_{\max}\right)\right]\end{aligned}$$

Therefore, a higher dispersion in portfolio choice leads to a higher dispersion of inequality. ■

Proof. Proof of Lemma 10 Direct calculation. We use 1 to denote a positive sign.

We have

$$\begin{aligned}& sign\left(\frac{\partial I\left(x\right)}{\partial d}\right)\\ =\; & -sign\left(\frac{\partial I\left(x\right)}{\partial\sigma^{2}\left(x\right)}\right)\\ =\; & -sign\left[\begin{array}{c}\frac{\alpha}{\gamma\sigma^{4}(x)}\left[-\left(1-\frac{(\gamma-1)\left(r_{f}-f_{xx}\right)+\rho}{\gamma}\right)+\frac{(\gamma-1)\alpha^{2}}{\gamma^{2}\sigma^{2}(x)}\right]Z\left(x\right)\\ +\frac{\alpha}{\gamma\sigma^{2}(x)}\left[1-\frac{(\gamma-1)\left(r_{f}-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^{2}}{2\gamma^{2}\sigma^{2}(x)}\right]z_{\min}\frac{-1}{\left[C_{1}\frac{\sigma^{2}(x)}{\alpha^{2}}-\gamma-1\right]^{2}}\frac{C_{1}}{\alpha^{2}}\end{array}\right]\\ \geq\; & -sign\left[-\left(1-\frac{\left(\gamma-1\right)\left(r_{f}-f_{xx}\right)+\rho}{\gamma}\right)+\frac{\left(\gamma-1\right)\alpha^{2}}{\gamma^{2}\sigma^{2}\left(x\right)}\right]\\ =\; & 1\end{aligned}$$

Second, for each level of expertise, we have

$$
\begin{aligned}
sign\left(\frac{\partial I(x)}{\alpha}\right) &= sign\left[\begin{array}{c} \frac{1}{\gamma\sigma^2(x)}\frac{-(\gamma-1)\left(r_f-f_{xx}\right)+\gamma-\rho}{\gamma}Z(x)-\frac{1}{\gamma\sigma^2(x)}\frac{3(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}Z(x) \\ +\frac{\alpha}{\gamma\sigma^2(x)}\left[\frac{-(\gamma-1)\left(r_f-f_{xx}\right)+\gamma-\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]z_{\min}\frac{2C_1\frac{\sigma^2(x)}{\alpha^3}}{\left[C_1\frac{\sigma^2(x)}{\alpha^2}-\gamma-1\right]^2} \end{array}\right] \\
&= sign\left[\begin{array}{c} 1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{3(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)} \\ +\left[\frac{-(\gamma-1)\left(r_f-f_{xx}\right)+\gamma-\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]\frac{z_{\min}}{Z(x)}\frac{2C_1\frac{\sigma^2(x)}{\alpha^2}}{\left[C_1\frac{\sigma^2(x)}{\alpha^2}-\gamma-1\right]^2} \end{array}\right] \\
&\geq sign\left[1-\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}-\frac{3(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right] \\
&\geq sign\left[1-\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}-\frac{3(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(\bar{x})}\right] \\
&= 1,
\end{aligned}
$$

Third, for each level of expertise, we have

$$
\begin{aligned}
& sign\left(\frac{\partial I(x)}{\partial\sigma_\upsilon}\right) \\
&= sign\left(\frac{\partial I(x)}{\partial\sigma^2(x)}\frac{\partial\sigma^2(x)}{\partial\sigma_\upsilon}\right) \\
&= sign\left(\frac{\partial I(x)}{\partial\sigma^2(x)}\right)sign\left(\frac{\partial\sigma^2(x)}{\partial\sigma_\upsilon}\right) \\
&= -1.
\end{aligned}
$$

Fourth, for each level of expertise:

$$
\begin{aligned}
sign\left(\frac{\partial I(x)}{\partial\gamma}\right) &= sign\left[\begin{array}{c} -\frac{1}{\gamma^2}\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]Z(x) \\ +\frac{1}{\gamma}\left[-\frac{r_f-f_{xx}}{\gamma^2}+\frac{\rho}{\gamma^2}-\frac{-\gamma+2}{2\gamma^3}\frac{\alpha^2}{\sigma^2(x)}\right]Z(x) \\ -\frac{1}{\gamma}\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right]z_{\min}\frac{1}{[\beta-1]^2}\frac{\beta}{\gamma} \end{array}\right] \\
&= sign\left[\begin{array}{c} -\left[1-\frac{(\gamma-2)\left(r_f-f_{xx}\right)}{\gamma}-\frac{2\rho}{\gamma}+\frac{-2\gamma+3}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right] \\ -\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right]\frac{1}{\beta-1} \end{array}\right] \\
&\leq sign\left[-\left(1-\frac{(\gamma-2)(r_f-f_{xx})+2\rho}{\gamma}-\frac{2\gamma-3}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right)\right] \\
&\leq sign\left[-1+\frac{2(\gamma-1)(r_f-f_{xx})+2\rho}{\gamma}+\frac{2\gamma-2}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right] \\
&= -1
\end{aligned}
$$

Lastly, for each level of expertise:

$$\begin{aligned}
sign\left(\frac{\partial I(x)}{\partial f_{xx}}\right) &= sign\left[\begin{array}{c} \frac{\alpha}{\gamma\sigma^2(x)}\frac{(\gamma-1)}{\gamma}Z(x) \\ -\frac{\alpha}{\gamma\sigma^2(x)}\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]z_{\min}\frac{2\gamma}{[\beta-1]^2}\frac{\sigma^2(x)}{\alpha^2} \end{array}\right] \\
&= sign\left[\begin{array}{c} \frac{(\gamma-1)}{2\gamma}\frac{\alpha^2}{\sigma^2(x)} \\ -\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]\frac{1}{\beta(\beta-1)} \end{array}\right] \\
&= -sign\left[1-\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}(1+\beta(\beta-1))\right] \\
&= -1 \text{ if } y^x(\bar{x}) < \frac{1}{1+\beta(\beta-1)}
\end{aligned}$$

■

Weaker conditions for Proposition 11 Some weaker conditions are:

$$\begin{aligned}
\underbrace{1-\frac{(\gamma-1)\rho_f+\rho}{\gamma}}_{\text{investment of non-expert}}+\underbrace{\frac{(\gamma-1)f_{xx}}{\gamma}}_{\text{cost of being expert}}-\underbrace{\frac{3(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(\bar{x})}}_{\text{benefit of being expert}} &> 0, \\
or\ y^x(\bar{x}) &< 1-\frac{(\gamma-1)}{\gamma^2}\frac{\alpha^2}{\sigma^2(\bar{x})} \\
or\ y^x(\bar{x}) &< \frac{1}{3}\left[1+\frac{2(\gamma-1)(r_f-f_{xx})+2\rho}{\gamma}\right] \\
or\ \frac{\alpha^2}{\sigma^2(\bar{x})} &< \frac{2\gamma^2}{3(\gamma-1)}\left[1-\frac{(\gamma-1)(\rho_f-f_{xx})+\rho}{\gamma}\right]
\end{aligned}$$

Proof. Proof of Proposition 11 For each level of expertise, we have

$$
\begin{aligned}
sign\left(\frac{\partial I(x)}{\alpha}\right) &= sign\left[\begin{array}{c} \frac{1}{\gamma\sigma^2(x)}\frac{-(\gamma-1)\left(r_f-f_{xx}\right)+\gamma-\rho}{\gamma}Z(x)-\frac{1}{\gamma\sigma^2(x)}\frac{3(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}Z(x) \\ +\frac{\alpha}{\gamma\sigma^2(x)}\left[\frac{-(\gamma-1)\left(r_f-f_{xx}\right)+\gamma-\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]z_{\min}\frac{2C_1\frac{\sigma^2(x)}{\alpha^3}}{\left[C_1\frac{\sigma^2(x)}{\alpha^2}-\gamma-1\right]^2} \end{array}\right] \\
&= sign\left[\begin{array}{c} 1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{3(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)} \\ +\left[\frac{-(\gamma-1)\left(r_f-f_{xx}\right)+\gamma-\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]\frac{z_{\min}}{Z(x)}\frac{2C_1\frac{\sigma^2(x)}{\alpha^2}}{\left[C_1\frac{\sigma^2(x)}{\alpha^2}-\gamma-1\right]^2} \end{array}\right] \\
&\geq sign\left[1-\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}-\frac{3(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right] \\
&\geq sign\left[1-\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}-\frac{3(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(\bar{x})}\right] \\
&= 1,\ \text{for all } x \text{ such that } \frac{\alpha^2}{2\sigma^2(x)\gamma}\geq f_{xx}
\end{aligned}
$$

And when α is higher, more experts enter. Thus

$$\frac{\partial I}{\partial \alpha} > 0.$$

■

Weaker conditions for Proposition 12 Some weaker conditions are:

$$
\begin{aligned}
1-\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{\gamma^2\sigma^2(x)} &> 0, \\
or\ y^x(\bar{x}) &< 1-\frac{(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(\bar{x})} \\
or\ y^x(\bar{x}) &< \frac{1}{2}\left[1+\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}\right] \\
or\ \frac{\alpha^2}{\sigma^2(\bar{x})} &< \frac{\gamma^2}{\gamma-1}\left[1-\frac{(\gamma-1)(r-f_{xx})+\rho}{\gamma}\right]
\end{aligned}
$$

Proof. Proof of Proposition 12 Direct calculation. We use 1 to denote a positive sign.

$$\begin{aligned} & sign\left(\frac{\partial I\left(x\right)}{\partial \sigma_v}\right) \\ = \ & sign\left(\frac{\partial I\left(x\right)}{\partial \sigma^2\left(x\right)}\frac{\partial \sigma^2\left(x\right)}{\partial \sigma_v}\right) \\ = \ & sign\left(\frac{\partial I\left(x\right)}{\partial \sigma^2\left(x\right)}\right) sign\left(\frac{\partial \sigma^2\left(x\right)}{\partial \sigma_v}\right). \end{aligned}$$

We also have

$$\begin{aligned} & -sign\left(\frac{\partial I\left(x\right)}{\partial d}\right) \\ = \ & sign\left(\frac{\partial I\left(x\right)}{\partial \sigma^2\left(x\right)}\right) \\ = \ & sign\left[\begin{array}{c} \frac{\alpha}{\gamma\sigma^4(x)}\left[-\left(1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}\right)+\frac{(\gamma-1)\alpha^2}{\gamma^2\sigma^2(x)}\right] Z\left(x\right) \\ +\frac{\alpha}{\gamma\sigma^2(x)}\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right] z_{\min}\frac{-1}{\left[C_1\frac{\sigma^2(x)}{\alpha^2}-\gamma-1\right]^2}\frac{C_1}{\alpha^2} \end{array}\right] \\ \leq \ & sign\left[-\left(1-\frac{\left(\gamma-1\right)\left(r_f-f_{xx}\right)+\rho}{\gamma}\right)+\frac{\left(\gamma-1\right)\alpha^2}{\gamma^2\sigma^2\left(x\right)}\right] \\ = \ & -1 \end{aligned}$$

Thus for each level of expertise, when fundamental risk is higher, the demand for the complex risky asset is smaller. And when σ_v is higher, fewer experts enter the complex risky asset market. Thus

$$\frac{\partial I}{\partial \sigma_v} < 0.$$

Next, for each level of expertise:

$$
\begin{aligned}
sign\left(\frac{\partial I(x)}{\partial \gamma}\right) &= sign\begin{bmatrix} -\frac{1}{\gamma^2}\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]Z(x) \\ +\frac{1}{\gamma}\left[-\frac{r_f-f_{xx}}{\gamma^2}+\frac{\rho}{\gamma^2}-\frac{-\gamma+2}{2\gamma^3}\frac{\alpha^2}{\sigma^2(x)}\right]Z(x) \\ -\frac{1}{\gamma}\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right]z_{\min}\frac{1}{[\beta-1]^2}\frac{\beta}{\gamma} \end{bmatrix} \\
&= sign\begin{bmatrix} -\left[1-\frac{(\gamma-2)\left(r_f-f_{xx}\right)}{\gamma}-\frac{2\rho}{\gamma}+\frac{-2\gamma+3}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right] \\ -\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right]\frac{1}{\beta-1} \end{bmatrix} \\
&\leq sign\left[-\left(1-\frac{(\gamma-2)(r_f-f_{xx})+2\rho}{\gamma}-\frac{2\gamma-3}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right)\right] \\
&\leq sign\left[-1+\frac{2(\gamma-1)(r_f-f_{xx})+2\rho}{\gamma}+\frac{2\gamma-2}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right] \\
&= -1
\end{aligned}
$$

Lastly, for each level of expertise:

$$
\begin{aligned}
sign\left(\frac{\partial I(x)}{\partial f_{xx}}\right) &= sign\begin{bmatrix} \frac{\alpha}{\gamma\sigma^2(x)}\frac{(\gamma-1)}{\gamma}Z(x) \\ -\frac{\alpha}{\gamma\sigma^2(x)}\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]z_{\min}\frac{2\gamma}{[\beta-1]^2}\frac{\sigma^2(x)}{\alpha^2} \end{bmatrix} \\
&= sign\begin{bmatrix} \frac{(\gamma-1)}{2\gamma}\frac{\alpha^2}{\sigma^2(x)} \\ -\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]\frac{1}{\beta(\beta-1)} \end{bmatrix} \\
&= -sign\left[1-\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}(1+\beta(\beta-1))\right] \\
&= -1 \text{ if } y^x(\bar{x}) < \frac{1}{1+\beta(\beta-1)}
\end{aligned}
$$

Therefore:

$$\frac{\partial I}{\partial \gamma} < 0 \text{ and } \frac{\partial I}{\partial f_{xx}} < 0$$

■

Proof. Proof of Proposition 13 We have

$$sign\left(\frac{\partial I(x)}{\partial x}\right) = sign\left(\frac{\partial I(x)}{\partial d}\frac{\partial d}{\partial x}\right) = sign\left(-\frac{\partial I(x)}{\partial d}\frac{\partial \sigma(x)}{\partial x}\right) = 1$$

And

$$\begin{aligned} I(\Lambda_1) - I(\Lambda_2) &= \int [\lambda_1(x) - \lambda_2(x)] I(x)\, dx \\ &= -I(x)[\Lambda_1(\underline{x}) - \Lambda_2(\underline{x})] - \int \frac{\partial I(x)}{\partial x} [\Lambda_1(x) - \Lambda_2(x)]\, dx \\ &> 0 \end{aligned}$$

■

Value Weighed Equilibrium Sharpe ratio The market value weighted Sharpe ratio can be written as

$$\begin{aligned} SR^{vw} &= E\left[\frac{\theta(z-c)}{I}\frac{\alpha}{\sigma(x)} | \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right] \\ &= E\left[\frac{\theta z\left(1 - [y^x(x)]^{-\frac{1}{\gamma}}\right)}{I}\frac{\alpha}{\sigma(x)} | \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right] \\ &= E\left[\frac{\alpha}{\gamma\sigma^2(x)}\frac{1}{I}\frac{\alpha}{\sigma(x)}\epsilon^x\left(Z(x) | \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right)\right] \\ &= \frac{\alpha}{\gamma I}E\left[\frac{1 - [y^x(x)]^{-\frac{1}{\gamma}}}{\sigma^3(x)} Z(x) | \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right] \end{aligned}$$

Proof. Proof of Proposition 15 We have

$$sign\left[\frac{\partial I(x)}{\partial a}\right] = sign\left[\frac{\partial I(x)}{\partial \sigma^2(x)}\frac{\partial \sigma^2(x)}{\partial a}\right] = 1$$

And,

$$sign\left(\frac{\partial \lambda}{\partial \mu_x}\right) = sign\left(\log x - \mu_x + \frac{1}{2\sigma_x^2}\right) = 1$$

Lastly,

$$\begin{aligned} \frac{\partial \lambda}{\partial \sigma_x} &= -\frac{1}{x\sigma_x^2\sqrt{2\pi}}\exp\left[-\frac{\left(\log x - \mu_x + \frac{1}{2\sigma_x^2}\right)^2}{2\sigma_x^2}\right] \\ &\quad + \frac{1}{x\sigma_x\sqrt{2\pi}}\exp\left[-\frac{\left(\log x - \mu_x + \frac{1}{2\sigma_x^2}\right)^2}{2\sigma_x^2}\right]\left[\frac{\left(\log x - \mu_x + \frac{1}{2\sigma_x^2}\right)^2}{\sigma_x^3} + \frac{\left(\log x - \mu_x + \frac{1}{2\sigma_x^2}\right)}{\sigma_x^2}\frac{1}{\sigma_x^3}\right] \end{aligned}$$

Thus,

$$\begin{aligned}
sign\left(\frac{\partial\lambda}{\partial\sigma_x}\right) &= sign\left[-1+\frac{\left(\log x-\mu_x+\frac{1}{2\sigma_x^2}\right)^2}{\sigma_x^2}+\frac{\left(\log x-\mu_x+\frac{1}{2\sigma_x^2}\right)}{\sigma_x^2}\frac{1}{\sigma_x^2}\right] \\
&= sign\left[-\sigma_x^2+\left(\log x-\mu_x+\frac{3}{2\sigma_x^2}\right)\left(\log x-\mu_x+\frac{1}{2\sigma_x^2}\right)\right] \\
&\geq sign\left[-\sigma_x^2+\left(\log x-\mu_x+\frac{1}{2\sigma_x^2}\right)^2\right] \\
&= sign\left[\left(\log x-\mu_x+\frac{1}{2\sigma_x^2}-\sigma_x\right)\left(\log x-\mu_x+\frac{1}{2\sigma_x^2}+\sigma_x\right)\right] \\
&= 1
\end{aligned}$$

■

Proof. Proof of Lemma 16 First, we have

$$\begin{aligned}
& E\left[\sigma^2(x)\,|\sigma^2(x)\leq\frac{\alpha^2}{2\gamma f_{xx}}\right] \\
=\ & \int_{d\leq\frac{\alpha^2}{2\gamma f_{xx}}} xg(d)\,dx \\
=\ & E\left[\sigma^2(x)\right]-E\left[\sigma^2(x)\,|\sigma^2(x)>\frac{\alpha^2}{2\gamma f_{xx}}\right] \\
=\ & \underbrace{\left[a+\exp\left(-b\left(\mu_x-\frac{\sigma_x^2}{2}\right)-\log\sigma^2+\frac{1}{2}b^2\sigma_x^2\right)\right]}_{\text{unconditional mean of volatility}}\underbrace{\Phi\left(\frac{b\left(\mu_x-\frac{\sigma_x^2}{2}\right)+\log\sigma^2-b^2\sigma_x^2+\log\frac{\alpha^2}{\sigma^2 2\gamma f_{xx}}}{b\sigma_x}\right)}_{\text{number of experts}}.
\end{aligned}$$

Then direct calculations give us the above results. ■

Proof. Proof of Proposition 17 First, we have

$$\frac{\sigma_\upsilon}{\sigma(x)}\sim\log N\left(\frac{b}{2}\left(\mu_x-\frac{\sigma_x^2}{2}\right),\frac{b^2\sigma_x^2}{4}\right)\text{ with support }x\in\left(0,\frac{1}{a}\right),$$

$$
\begin{aligned}
& \mathbb{E}\left[\frac{\alpha}{\sigma(x)}|\sigma^2(x) \leq \frac{\alpha^2}{2\gamma f_{xx}}\right] \\
= \ & \frac{\alpha}{\sigma_v}\mathbb{E}\left[\frac{\sigma_v}{\sigma(x)}|\sigma^2(e) \leq \frac{\alpha^2}{2\gamma f_{xx}}\right] \\
= \ & \frac{\alpha}{\sigma_v}\mathbb{E}\left[\frac{\sigma_v}{\sigma(x)}|\frac{\sigma_v}{\sigma(x)} \geq \frac{\sigma_v\sqrt{2\gamma f_{xx}}}{\alpha}\right]
\end{aligned}
$$

Direction calculation implies that the expectation is higher when 1) a is smaller, 2) σ_v is smaller, 3) α is larger. ■

Proof. Proof of Proposition 14. Given

$$
\frac{\partial SR(x)}{\partial \sigma_v} = \frac{\frac{\partial \alpha}{\partial \sigma_v}\sigma(x) - \alpha\frac{\partial \sigma(x)}{\partial \sigma_v}}{\sigma^2(x)}
$$

we have

$$
\frac{\partial SR(x)}{\partial \sigma_v} > 0 \text{ iff } \frac{\partial \log \sigma(x)}{\partial \log \sigma_v} < \frac{\partial \log \alpha}{\partial \log \sigma_v}.
$$

If $\frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ is a constant, we must have either $\frac{\partial \log \alpha}{\partial \log \sigma_v} > \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ for all x or $\frac{\partial \log \alpha}{\partial \log \sigma_v} < \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ for all x.

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} < 0$, and assume there is a cutoff x^* such that

$$
\frac{\partial \log \sigma(x^*)}{\partial \log \sigma_v} = \frac{\partial \log \alpha}{\partial \log \sigma_v},
$$

then for all $x < x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_v} < 0$; and for all $x > x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_v} > 0$.

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} > 0$, and assume there is a cutoff x^* such that

$$
\frac{\partial \log \sigma(x^*)}{\partial \log \sigma_v} = \frac{\partial \log \alpha}{\partial \log \sigma_v},
$$

then for all $x < x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_v} > 0$; and for all $x > x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_v} < 0$. ■