Soft Collateral, Bank Lending, and the Optimal Credit Rating System*

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Abstract

We study the optimal credit rating system in a general equilibrium setting where borrowers have incentives to renege on debt repayments. We show that credit exclusion creates "soft"collateral in the form of a borrower's reputation. Compared with individual lending, bank lending reduces search frictions, and thereby increases the cost of credit exclusion, boosts the value of soft collateral, and facilitates borrowing and lending. A dynamic rating system allows agents' ratings to migrate over time and fine-tunes agents' incentives. This reduces agency costs, makes better use of soft collateral, and improves social welfare. We show that the optimal rating system is coarse, as we observe in the real world.

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1 Introduction

Credit ratings evaluate the creditworthiness of a potential borrower—the likelihood that a person, corporation, local government, or sovereign country may default on its debt obligations. However, there are two aspects to creditworthiness: a borrower may default because it cannot repay its debts, or it may default because it is able but unwilling to repay its debts. To be useful, credit ratings must capture both possibilities.

Although most studies of credit ratings focus on assessing a borrower's innate ability to repay, our paper is part of the smaller literature that focuses on how ratings affect a borrower's willingness to repay—that is, its propensity to commit moral hazard. In this setting, the credit rating is not a passive signal of borrower "type," but rather a form of soft collateral that incentivize the borrower to repay debt obligations in the future. In this respect, it resembles eBay's ratings scheme:

> *"eBay, on the other hand, is an example of a reputation mechanism that primarily acts as a sanctioning device. eBay users do not rate sellers on the absolute quality of their products, but rather on how well they were able to deliver what was promised on the item description. The role of eBay's reputation mechanism is to promote honest trade rather than to distinguish sellers who sell high-quality products from those who sell low-quality products." (Dellarocas, 2005, page 10.)*[1]

Our focus on pure moral hazard produces five key predictions, many of which are more in line with real world rating systems and credit markets than those of earlier models. First, we predict that, in equilibrium, there is a meaningful cross-sectional distribution of credit ratings at any one time. Second, contrary to the monotonic convergence of reputation predicted by many other papers, we find that borrowers' ratings can migrate over time. Third,

[1] eBay's system allows bad reviews to be dropped after a certain period of time has passed, which is analogous to our allowing reputations to be restored with a certain probability. A similar theme is found in auto driver point systems, which allow drivers several chances of vialations before suspending or revoking their licences, and, in addition, provides that violations are dropped after a certain amount of time has passed without incident. The rationale is that stripping away people's driving privilege is socially costly, but is necessary measure to deter bad driving; the point system helps keep this cost to a minimum.

we show that, in our moral hazard setting, loan rates differentiated by rating cannot in themselves create sufficient discipline to prevent moral hazard; successful discipline requires some possibility of credit exclusion for defaulting borrowers. Fourth, it is not optimal to exclude defaulting borrowers permanently; instead, as in the real world, it is optimal for excluded borrowers to eventually return to the credit market. Finally, we show that, consistent with real rating systems, the optimal rating system is "coarse" in the sense that it consists of a finite number of rating "grades."

More specifically, we analyze a setting where potential borrowers differ observably in their expected productivity, but the actual success or failure of a borrower's project cannot be observed. This gives borrowers incentives to default strategically ex post. We show that punishing defaulters with future exclusion from the credit market creates incentives for borrowers to behave. In effect, a reputation for not defaulting creates soft collateral that is required for continued credit access in the future. Moreover, as the financial system develops, the efficiency of this soft collateral mechanism improves, as we now discuss.

In a decentralized market with individual lending, it is difficult for borrowers and lenders to link up; hence future credit exclusion imposes a lower effective cost on defaulters—essentially, there are fewer future profits to be lost. This means that maintaining incentives against strategic default in such a market requires high probabilities of exclusion on default and low probabilities that excluded borrowers are later allowed to return. Because exclusion reduces potentially productive investment, and some defaults are unavoidable due to bad project outcomes, higher levels of exclusion reduce welfare relative to the first best.

By contrast, a banking market reduces search frictions by providing centralized intermediaries that borrowers and lenders can interact with. This form of financial development improves welfare in two ways. Obviously, it directly improves financing efficiency. Less obviously, by improving financing efficiency, the banking system increases the opportunity cost of credit exclusion and consequently raises the value of soft collateral, making it possible to deter strategic default with a lower likelihood and a shorter length of credit exclusion.

Further improvement can arise if banks use a multi-tiered credit rating scheme. Effectively, the added gradations function as multiple levels of soft collateral, and only the defaulters with the lowest rating—those without sufficient soft collateral—need to suffer credit exclusion in order to deter strategic default by all borrowers. Nevertheless, adding tiers to the credit rating system is not without cost: although defaulters are excluded less frequently, maintaining incentives requires that it be harder for defaulters to return to the credit market once they have been excluded. As a result, the optimal rating system does not simply maximize the number of rating tiers; instead, it balances the frequency and severity of punishment and minimizes the expected social cost.[2]

Throughout our analysis, we focus on general equilibrium in the steady state, where the number of people that are newly excluded from the credit market must balance the number of people allowed to return to the market. Among other things, this implies that any equilibrium with lending cannot punish defaulters indefinitely (the so-called "grim trigger" strategy); otherwise in the long run the number of agents still allowed to borrow would fall to zero. It follows that exclusion must allow some chance of reinstatement if lending is to prevail in the long run—which is precisely what we see in many countries, including the U.S.

As we mentioned earlier, our paper is part of an earlier literature that examines credit ratings and their effect on borrower moral hazard. This literature begins with Diamond (1989), who shows that reputation can be used to alleviate the conflict of interest between borrowers and lenders in a model with three types of borrowers—those who are innately good, those who are innately bad, and those who can choose to be good or bad. In his model, as time goes on, innately bad borrowers default and drop out of the cohort, improving the reputation of the borrowers that remain; this in turn can give strategic borrowers more incentives to choose good projects (though in the long-run they eventually "harvest" their improved reputation and choose the bad project). By contrast, Vercammen (1995) shows

[2] We emphasize that, in our model, probabilities of credit exclusion and reinstatement are imposed by a central planner/credit registry. Analyzing the incentives of individual banks or competing agencies in assigning ratings would be interesting but is left to future research.

that if bad borrowers are never excluded from the market, then the reputation effect can decrease over time as lenders learn more and more about borrowers' types. Finally, Padilla and Pagano (2000) show that, in a two-period model, information sharing between banks can mitigate moral hazard in effort provision: to avoid being pooled with low-quality borrowers, high-quality borrowers work hard to avoid default.[3]

In addition to work on how ratings affect borrower moral hazard, a more recent and rapidly-expanding literature focuses on moral hazard on the part of the rating agencies themselves. In these models, the borrower usually does not have a capacity for moral hazard, but there is a borrower adverse selection problem which the ratings agency can choose to overcome. Salient examples include Mathis, McAndrews, and Rochet (2009), Bolton, Freixas, and Shapiro (2012), Bar-Isaac and Shapiro (2013), Opp, Opp, and Harris (2013), Fulghieri, Strobl, and Xia (2014), Skreta and Veldkamp (2009), and Sangiorgi and Spatt (2015). Boot, Milbourn, and Schmeits (2006) are somewhat closer to our focus: in their model, the threat of a potential ratings downgrade can deter borrower moral hazard.

Our paper is more closely related to work on ratings coarseness. Lizzeri (1999) shows that, in order to maximize surplus, a monopoly intermediary has incentive to manipulate information by revealing only whether quality is above some minimal standard. By contrast, competition among intermediaries can force them to reveal full information. Goel and Thakor (2013) construct a cheap-talk game to model coarse ratings. In equilibrium, a rating agency wants to deliver inflated ratings to please issuers, and, in the meantime, needs to keep the rating inflation below a threshold to make it credible to investors. The two conflicting objectives give rise to coarse but unbiased ratings in equilibrium. Coarse ratings reduce social welfare because they lead to investment inefficiency. Kovbasyuk (2013) shows that private rating-contingent payments can cause ratings coarseness. Kartasheva and Yilmaz (2013) show that ratings become less precise when there are more uninformed investors in the market and the gains of trade increase. Donaldson and Piacentino (2013) consider credit

[3]For examples of the broader literature focusing on credit ratings when borrower quality varies but moral hazard is not present, see Diamond (1991), Pagano and Jappelli (1993), and Padilla and Pagano (1997).

ratings as a source of public information and show that a reduction in rating precision can Pareto improve social welfare. Our paper is different in that, instead of considering rating agencies' incentives and the relative advantage of private information, we focus on the effect of ratings on borrowers' incentives. An optimal rating system has to be coarse because it needs to satisfy incentive compatibility constraints of agents with various ratings.

Our model is also related to research on dynamic contracting with moral hazard. Gromb (1999) studies a multiperiod model where withholding future funding is a threat to deter strategic default. He shows that renegotiation can erode the lender's profit, sometimes to the point that lending collapses. In a discrete-time setting, DeMarzo and Fishman (2007) characterize the optimal dynamic contract in terms of payments and termination probability as functions of the agent's continuation payoff, and show that this can be implemented with debt, equity and a line of credit. DeMarzo and Sannikov (2006) extend this to a continuous-time setting, and Biais et al. (2007) use similar methods to examine the use of cash reserves in the optimal contract as well as the model's asset-pricing implications.[4]

Unlike these earlier papers on dynamic moral hazard, we analyze a general equilibrium model that incorporates the aggregate supply and demand of capital, the bargaining power of lenders and borrowers, and different types of financial system sophistication. To maintain a steady-state equilibrium, we allow excluded borrowers to have a chance of returning to the market in the future. The optimal rating system endogenously determines how quickly an excluded borrower should be allowed back to the market, which pins down a borrower's minimum payoff in equilibrium. If we view a rating system as a contract, then the optimal

[4]On a related front, Bond and Krishnamurthy (2004) study optimal enforcement mechanisms in a multiperiod setting where a borrower has no collateral and some form of exclusion from financial markets is needed to support lending from competitive banks. They show that a "debt-default" rule that prevents a defaulting borrower from placing assets in other banks before repaying any existing loans is enough to maintain the efficient outcome. Moreover, once complications such as lenders not being able to commit to make future loans or the enforcing authority having limited information are introduced, the debt-defuault rule remains efficient whereas other common proposals (complete exclusion from financial transactions or granting the lender monopoly power over the borrower) are not. By contrast, our paper does not focus on specific mechanisms that are needed to maintain coordination among lenders; instead, we assume the enforcement rule consists of financial exclusion and focus on the optimal probability and duration of exclusion after default.

contract in a general equilibrium setting differs from that in a partial equilibrium setting.[5,6] Our general equilibrium setting also allows us to examine the equilibrium distribution and migration of credit ratings, with predictions similar to what we observe in real-world consumer credit markets.[7]

The rest of the paper proceeds as follows. In Section 2, we set up the model and lay out the assumptions. In Section 3, we first study the autarky case where there is no borrowing and lending; we then analyze the credit market without banks where borrowing and lending can only occur through random matching of dispersed individuals. We examine the centralized bank lending market in Section 4. We investigate credit ratings in Section 5. We first study a simple three-tier rating system to illustration the intuition; afterwards we solve the general multi-tier rating equilibrium and characterize the optimal rating system. We discuss rate differentiation in Section 6. Section 7 discusses the empirical implications of our results in relation to stylized facts and concludes. Proofs of propositions may be found in the appendix.

2 Model

The economy is populated with a continuum of infinitely lived agents, with the total population normalized to unity. Agents produce and consume perishable goods at discrete points in continuous time. At the beginning of each period, agents receive two shocks: a capital endowment shock and a productivity shock. Specifically, a fraction $c \in [0, 1]$ of the population are each endowed with one (normalized) unit of capital, which is needed to produce consumption goods. In addition, all agents, with or without capital, receive a productivity

[5] In a partial equilibrium parincipal-agent setting, a higher probability of temporary termination combined with a certain revival probability (as in our model) is homomorphic to a lower probability of permanent termination (as in the dynamic contracting models). This is not true in general equibrium: if agents are not allowed to return to the capital market, the population of borrowers (and lending volume) would shrink to zero over time.

[6] Although our approach also differs in that we focus on an agent's credit rating rather than her continuation value, there is a one-to-one mapping between credit rating and continuation value.

[7] Two earlier papers on the equivalence between credit and money by Taub (1994) and Kocherlakota (1998) are also somewhat related to our paper, in that defaults on agreements are punished with autarky. Unlike our work, these papers focus on matching between individuals; they do not look at institutional improvements such as banks or more complex rating schemes.

shock that is independent of the capital endowment shock: with probability p, an agent's productivity is high (H); with probability $1-p$, his productivity is low (L). We assume that the distribution of capital endowment and productivity shocks are independent and identical across time.[8] So, conditional on capital endowment and productivity shocks, each period there are four types of agents in the economy: those with capital and high productivity, whose value function denoted by V_{1H}; those with capital but low productivity, whose value function denoted by V_{1L}; those with high productivity but no capital, whose value function denoted by V_{0H}; and those with low productivity and no capital, whose value function denoted by V_{0L}.

Capital cannot be consumed directly, but can be used to produce consumption goods that can be consumed at the end of the period. With one unit of capital, an agent with high productivity produces random output: either X units of the consumption good with probability π or zero consumption good with probability $1-\pi$; the expected output is $X_H = \pi X$. We assume that X_H is greater than a low-productivity agent's output per unit of capital, which, for simplicity, is assumed to be a positive constant $X_L > 0$. We assume that a high-productivity agent's realized output is neither observable nor verifiable, which gives rise to the moral hazard problem, the solution to which is the key point of the paper. We also assume capital goods are indivisible and each agent can only use one unit of capital. In addition, capital is perishable and fully depreciates at the end of a period, regardless of whether it has been used to produce consumption goods; hence, there is no capital accumulation.[9] All agents are risk neutral, and the discount rate is r per period. We first study the equilibrium in the absence of financial intermediaries.

[8] **Our main results still hold even with time persistent shocks except that the analysis would be more complicated because we would need a different set of value fuctions for agents at different states. For clarity, we focus our analysis on non-persistent shocks. However, it is worth pointing out that persistent shocks would make credit exclusion more costly for defaulters because they are more likely to need credit next period.**

[9] Even if an agent can only invest one unit of capital each period, he may have incentives to store capital as a precautionary measure against credit exclusion. For simplicity and tractability, we assume that capital is perishable and thus cannot be stored.

3 Equilibrium without Financial Intermediaries

In this section, we analyze the equilibrium in an economy where there is no financial intermediary. We first solve the autarky case, then consider the case where individual borrowing and lending are allowed.

3.1 Autarky

In the case of autarky, there is no borrowing and lending. The value functions are as follows:

$$
\begin{aligned}
V_{1H}^{A} &= \frac{1}{1+r}\{X_H + V^A\}, \\
V_{1L}^{A} &= \frac{1}{1+r}\{X_L + V^A\}, \\
V_{0H}^{A} &= \frac{V^A}{1+r}, \\
V_{0L}^{A} &= \frac{V^A}{1+r},
\end{aligned}
$$

where

$$
V^A \equiv cpV_{1H}^{A} + c(1-p)V_{1L}^{A} + (1-c)pV_{0H}^{A} + (1-c)(1-p)V_{0L}^{A}
$$

is the unconditional expected lifetime value at the beginning of a period, before agents learn the realizations of their capital and productivity shocks. The following proposition describes the autarky equilibrium:

Proposition 1 *In the autarky equilibrium, an agent's expected lifetime payoff is equal to* $\frac{cpX_H + c(1-p)X_L}{r}$.

Proof. See Appendix. ■

The proposition is easy to interpret. Each period an agent receives capital with probability c, and, with the capital endowment, produces X_H with probability p and X_L with probability $1-p$. Therefore, the expected payoff is $cpX_H + c(1-p)X_L$. The ex ante unconditional expected lifetime value, V^A, is just a perpetuity with the expected periodical payoffs equal to $cpX_H + c(1-p)X_L$. Ex post, if an agent does not own capital, he receives nothing during the current period, and thus the lifetime value is the perpetuity postponed by one period; discounted by $1+r$, it is $\frac{V^A}{1+r}$. If an agent owns capital in the current period, then in addition to the postponed perpetuity, he is going to receive X_H or X_L at the end of the current period depending on whether his productivity is high or low. Since agents are homogeneous, social welfare in the autarky economy is the same as an agent's unconditional expected lifetime value:

$$W^A = V^A = \frac{cpX_H + c(1-p)X_L}{r}.$$

The autarky economy is inefficient because a fraction of capital is stuck in the hands of those agents with low productivity while some of the high-productivity agents do not have access to the indispensable capital for the production of consumption goods. The inefficiency calls for a financial market where agents can borrow and lend capital to generate more outputs. In the remaining of this paper, we analyze financial markets that allow borrowing and lending, starting with individual loans, then bank loans, and finally, bank loans with credit ratings.

3.2 Individual Loans

In this section, we consider the case of a decentralized market with individual loans. We assume agents randomly meet after capital and productivity shocks are realized. Borrowing and lending happen only when a capital owner with low productivity meets an agent with no capital but high productivity; the former then becomes a capital borrower and the latter becomes a capital lender. Considering the overall distribution of different agent types, a borrower meets a lender with probability $c(1-p)$, and a lender meets a borrower with probability $(1-c)p$. A borrower agrees to pay R to the lender at the end of the period after production is completed.[10] Because production is risky, the lender has a chance to receive

[10] We assume that agents cannot pledge their future capital shocks. This assumption can be justified by agents' voluntary participation in the capital market: if an agent pledges too much of his future capital, then

R only when a high-productivity borrower generates X units of the consumption good; this happens with probability π. Moreover, because output is neither observable nor verifiable, without any potential punishment, the borrower has no incentive to repay the debt.

The punishment for default is credit exclusion. Specifically, we assume that with probability γ a defaulting borrower obtains a bad reputation and will be denied of loans from any other agent in the next period. However, reputation can be repaired. After one period, with probability η a defaulting agent will get a fresh start and be able to borrow again; with probability $1-\eta$, the bad reputation sticks and the defaulting agent has to wait for one more period to see whether he has a chance to be allowed to borrow. For now, we take these exclusion and reinstatement parameter values as given; we endogenize them in Section 5 below. In the steady state, a fraction α^I of the population do not have the bad reputation; their value functions conditional on realized capital and productivity shocks are as follows:

$$\begin{aligned}
V^I_{1H} &= \frac{1}{1+r}\{X_H + V^I\}, \\
V^I_{1L} &= \frac{1}{1+r}\{(1-c)p\pi R + (1-(1-c)p)X_L + V^I\}, \\
V^I_{0H} &= \frac{1}{1+r}\{c(1-p)[\pi(X-R+V^I) + (1-\pi)((1-\gamma)V^I + \gamma V^{I(e)})] + (1-c(1-p))V^I\}, \\
V^I_{0L} &= \frac{V^I}{1+r},
\end{aligned}$$

where

$$V^I \equiv cpV^I_{1H} + c(1-p)V^I_{1L} + (1-c)pV^I_{0H} + (1-c)(1-p)V^I_{0L}$$

is the unconditional expected lifetime value at the beginning of a period before capital and productivity shocks are realized. As for those agents with the bad reputation, the remaining $1-\alpha^I$ fraction of the population, they will be excluded from borrowing for at least one

he has the incentive to quit the capital market rather than make good on his promises.

period; we denote their unconditional expected lifetime value by $V^{I(e)}$:

$$V^{I(e)} = \frac{1}{1+r}[cpX_H + c(1-p)X_L + \eta V^I + (1-\eta)V^{I(e)}\}.$$

The equilibrium solutions of the value functions are subject to the following conditions:

1). Lenders are willing to lend:

$$\pi R \geq X_L;$$

2). Borrowers with high outputs are willing to repay the loan:

$$R \leq \gamma(V^I - V^{I(e)});$$

3). A constant steady state population distribution:

$$\alpha^I c(1-c)p(1-p)(1-\pi)\gamma = \eta(1-\alpha^I).$$

We assume that borrowers have all the bargaining power, so $R = X_L/\pi$.[11] A borrower chooses between repaying the loan and facing the punishment of potential credit exclusion next period. The threat of credit exclusion essentially serves as a form of soft collateral with value equal to $\gamma(V^I - V^{I(e)})$—that is, the opportunity cost of defaulting and being excluded from the credit market with probability γ. The likelihood of being excluded has a direct effect on this opportunity cost, and also has an indirect effect through its impact on $V^I - V^{I(e)}$. A borrower repays the debt if and only if the value of the soft collateral exceeds the gain from strategic default; i.e. $\gamma(V^I - V^{I(e)}) \geq R$. Solving the model, we have:

Proposition 2 *There exists a private loan market if and only if: 1) the likelihood of excluding a defaulting borrower from the capital market is large enough: given all the other*

[11] **If lenders get some of the surplus, then the loan rate goes up and credit exclusion becomes less costly, which means that we need a higher γ or a lower η to guarantee that the incentive compatibility condition is satisfied.**

parameters, there is a minimum value of γ, $\underline{\gamma}^I$, *such that* $\gamma \geq \underline{\gamma}^I$; *or 2) the chance of returning to the capital market is small enough: given all the other parameters, there is a maximum value of* η, $\overline{\eta^I}$, *such that* $\eta \leq \overline{\eta^I}$.

Proof. See Appendix. ■

Proposition 2 shows that the existence of a private loan market depends on the value of the soft collateral, which is determined by the likelihood of blackballing a defaulting borrower. Social welfare in this case is equal to the weighted average of the expected lifetime value:

$$\begin{aligned} W^I &= \alpha^I V^I + (1-\alpha^I)V^{I(e)} \\ &= \frac{cpX_H + c(1-p)X_L}{r} + \frac{c(1-c)p(1-p)(X_H - X_L)}{r[1 + c(1-c)p(1-p)(1-\pi)\gamma/\eta]}. \end{aligned}$$

As can be seen, social welfare is decreasing in γ and increasing in η. Being excluded from the capital market, defaulting borrowers cannot take advantage of their high productivity, but this welfare loss is the necessary cost to guarantee that borrowers have incentives to repay the debt.

The second incentive compatibility condition requires that the value of soft collateral is large enough to deter strategic default. A decentralized private loan market faces two obstacles that hamper the value of the soft collateral and hinder borrowing and lending. First, search frictions limit the chance of meeting a lender and thus soften the punishment of being excluded from the capital market. In addition, although we do not model it directly, it may be difficult to share information about a borrower's default and to exclude him in a decentralized setting, which would effectively make γ small and η large, making equilibrium harder to support. As a result, a private loan market can only exist in a closely-knit community where people are familiar with each other and information is relatively transparent. Even when a private loan market exists, the high exclusion needed to support equilibrium makes it extremely inefficient. We will now show that, for given parameter values of γ and η, a market intermediated by banks can boost the value of the soft collateral relative to a decentralized market, which in turn allows a welfare-improving reduction in the likelihood and duration of credit exclusion.

4 Banking

Suppose there is a competitive banking system in which banks accept deposits from agents who are endowed with capital and low productivity and make loans to agents who have high-productivity but lack capital. The existence of competitive banks alleviates the double coincidence problem because borrowers and depositors do business with banks instead of meeting each other through random matching. Improved access to finance makes it more costly for a borrower to default and be excluded from the capital market, boosting the value of soft collateral.

We assume that depositors receive R_d by saving their capital goods with banks; borrowers who receive bank loans agree to pay R_l at the end of the period. In the steady state, a fraction α^B of all the agents are allowed to borrow from banks and the remaining $1-\alpha^B$ of all the agents are excluded from borrowing for at least one period due to default in the past. Agents who are not blacklisted by banks have the following value functions once the capital and productivity shocks are realized:

$$
\begin{aligned}
V_{1H}^B &= \frac{1}{1+r}\{X_H + V^B\}, \\
V_{1L}^B &= \frac{1}{1+r}\{R_d + V^B\}, \\
V_{0H}^B &= \frac{1}{1+r}\{\pi(X - R_l + V^B) + (1-\pi)[(1-\gamma)V^B + \gamma V^{B(e)}]\}, \\
V_{0L}^B &= \frac{V^B}{1+r},
\end{aligned}
$$

where

$$
V^B \equiv cpV_{1H}^B + c(1-p)V_{1L}^B + (1-c)pV_{0H}^B + (1-c)(1-p)V_{0L}^B
$$

is the unconditional expected lifetime value at the beginning of a period. Agents who are blacklisted by banks have the following unconditional expected lifetime value function:

$$V^{B(e)} = \frac{1}{1+r}\{cpX_H + c(1-p)X_L + \eta V^B + (1-\eta)V^{B(e)}\}.$$

In equilibrium, the following constraints need to be satisfied:

1). Depositors are willing to put their capital into banks:

$$R_d \geq X_L;$$

2). Borrowers with high outputs are willing to repay bank loans:

$$R_l \leq \gamma(V^B - V^{B(e)});$$

3). Banks break even:

$$\pi R_l \geq R_d;$$

4). A constant steady state population distribution:

$$\alpha^B(1-c)p(1-\pi)\gamma = \eta(1-\alpha^B).$$

We still assume that the overall supply of deposits is greater than the demand for loans. As a result, banks will compete to lower the deposit rate and the loan rate such that we have $R_d = X_L$ and $\pi R_l = R_d$ in equilibrium. Proposition 3 characterizes the equilibrium solutions:

Proposition 3 *Compared with the private loan market, a competitive banking system improves economic efficiency. Specifically, let* $\underline{\gamma}^B$ *(* $\overline{\eta^B}$*) denote the minimum (maximum) value of* γ *(*η*), ceteris paribus, for a bank loan equilibrium to exist. We have* $\underline{\gamma}^B < \underline{\gamma}^I$ *and* $\overline{\eta^B} > \overline{\eta^I}$*; that is, when* $\underline{\gamma}^B \leq \gamma < \underline{\gamma}^I$ *(or* $\overline{\eta^B} \geq \eta > \overline{\eta^I}$*), there exists a bank loan equilibrium but not a* *private loan equilibrium. In addition, when $\gamma \geq \underline{\gamma^I}$ (or $\eta \leq \overline{\eta^I}$), a bank loan equilibrium is always more efficient than a private loan equilibrium.*

Proof. See Appendix ■

With banks present in the economy, borrowers know where exactly to obtain capital to exploit their high productivity and will always get it if they are not blacklisted by banks. In contrast, because of search frictions in the private loan economy, a borrower with good reputation only obtains capital with probability $c(1-p)$–the agent he meets is endowed with capital and low productivity. Proposition 3 shows that the reduction of search frictions has a huge impact beyond itself because it greatly increases the cost of credit exclusion, and, by doing so, it increases the value of soft collateral. Consequently, a small chance of being blacklisted by banks can become a huge cost for defaulting borrowers. Through this channel, a centralized loan market tightens borrowers' incentives, relaxes constraints on parameters, and improves social welfare.

What is worth mentioning is that, although concentrated lending makes it easier to blacklist defaulting borrowers–that is, bank lending is presumably associated with a higher γ and a lower η, this is not the source of improved efficiency; instead, if anything, it is a source of inefficiency. We only need the parameter values of γ and η to guarantee the existence of the bank loan equilibrium; beyond those values, a higher γ or a lower η reduces social welfare.

So far we have shown that a competitive banking system is more efficient than a private loan economy, but can it be further improved? We will now show that a more complex system with multiple levels of ratings may improve matters over the simple exclusion-reinstatement schemes we have been analyzing. Intuitively, such a rating system stratifies agents into groups with different levels of reputation (soft collateral). This allows a multiple-tier punishment scheme in which defaulting borrowers with sufficiently high ratings (soft collateral) only lose part of this collateral by being downgraded instead of being immediately excluded from the capital market; only those defaulting borrowers with very low ratings (and thus insufficient soft collateral) are actually excluded. A properly designed ratings system can maintain borrowers incentives to repay their loans and repair their ratings while requiring an expected probability and duration of exclusion that is lower than in the simple scheme used in the previous sections—i.e., rating downgrades may be a less costly solution to the moral hazard problem. We investigate credit ratings in the next section.

5 Credit Ratings

To understand how a rating system contributes to social welfare, we first analyze a simple case where each agent is assigned one of the three ratings: A, B, or C. Afterwards, we extend our analysis to a general system with N ratings and characterize the optimal rating system.

5.1 A Three-tier Rating System

We extend the analysis in Section 4 by further dividing those agents who are not excluded from borrowing into two subgroups: A and B. So at the beginning of each period, before capital and productivity shocks are realized, each agent has one of the three ratings: A, B, or C; C means exclusion. If an agent with rating A borrows and defaults, then his rating is downgraded to B; otherwise he keeps the original rating A. If an agent with rating B borrows and repays the loan, his rating is upgraded to A; if he borrows and defaults, then his rating is downgraded to C with probability γ; in all other cases he keeps the original rating B. An agent with rating C is excluded from borrowing in the current period but has a chance to be upgraded to rating B next period, which happens with probability η; with probability $1-\eta$, he remains the original rating C next period. We use superscripts RA, RB, and RC to differentiate agents with ratings A, B, and C respectively.[12]

Agents with rating A have the following value functions once the capital and productivity

[12] We assume that the rating agencys and banks commit not to renegotiating agents' ratings. Individual banks may have incentives to renegotiate with borrowers, but renegotiation unravels the credit rating system.

shocks are realized:

$$
\begin{aligned}
V_{1H}^{RA} &= \frac{1}{1+r}\{X_H + V^{RA}\}, \\
V_{1L}^{RA} &= \frac{1}{1+r}\{R_d + V^{RA}\}, \\
V_{0H}^{RA} &= \frac{1}{1+r}\{\pi(X - R_l + V^{RA}) + (1-\pi)V^{RB}]\}, \\
V_{0L}^{RA} &= \frac{V^{RA}}{1+r},
\end{aligned}
$$

where

$$V^{RA} \equiv cpV_{1H}^{RA} + c(1-p)V_{1L}^{RA} + (1-c)pV_{0H}^{RA} + (1-c)(1-p)V_{0L}^{RA}$$

is the unconditional expected lifetime value at the beginning of a period. Agents with rating B have the following value functions once the capital and productivity shocks are realized:

$$
\begin{aligned}
V_{1H}^{RB} &= \frac{1}{1+r}\{X_H + V^{RB}\}, \\
V_{1L}^{RB} &= \frac{1}{1+r}\{R_d + V^{RB}\}, \\
V_{0H}^{RB} &= \frac{1}{1+r}\{\pi(X - R_l + V^{RA}) + (1-\pi)[(1-\gamma)V^{RB} + \gamma V^{RC}]\}, \\
V_{0L}^{RB} &= \frac{V^{RB}}{1+r},
\end{aligned}
$$

where

$$V^{RB} \equiv cpV_{1H}^{RB} + c(1-p)V_{1L}^{RB} + (1-c)pV_{0H}^{RB} + (1-c)(1-p)V_{0L}^{RB}$$

is the unconditional expected lifetime value at the beginning of a period.

Agents with rating C have the following unconditional expected lifetime value:

$$V^{RC} = \frac{1}{1+r}\{cpX_H + c(1-p)X_L + \eta V^{RB} + (1-\eta)V^{RC}\}.$$

The value functions are subject to the following constraints:

1). Depositors are willing to deposit their capital in banks:

$$R_d \geq X_L;$$

2a). Borrowers with rating A are willing to repay bank loans:

$$R_l \leq V^{RA} - V^{RB};$$

2b). Borrowers with rating B are willing to repay bank loans:

$$R_l \leq V^{RA} - [\gamma V^{RC} + (1-\gamma)V^{RB}];$$

3). Banks break even:

$$\pi R_l \geq R_d;$$

4). A constant steady state population distribution:

$$\begin{aligned} \alpha^{RA}(1-c)p(1-\pi) &= \alpha^{RB}(1-c)p\pi, \\ \alpha^{RB}(1-c)p(1-\pi)\gamma^B &= (1-\alpha^{RA}-\alpha^{RB})\eta^B, \end{aligned}$$

where α^{RA} and α^{RB} denote the proportion of agents with ratings A and B respectively.

Same as before, we assume that competition drives the deposit rate and the loan rate to the minimum level; that is, $R_d = X_L$ and $\pi R_l = X_L$.

Proposition 4 *Let $\underline{\gamma^R}$ ($\overline{\eta^R}$) denote the minimum (maximum) value of γ (η), ceteris paribus, for a bank loan equilibrium with credit ratings to exist. We have $\underline{\gamma^R} > \underline{\gamma^B}$ and $\overline{\eta^R} < \overline{\eta^B}$; that is, when $\underline{\gamma^B} \leq \gamma < \underline{\gamma^R}$ (or $\overline{\eta^B} \geq \eta > \overline{\eta^R}$), there exists a bank loan equilibrium without credit ratings but not a bank loan equilibrium with credit ratings. However, when $\gamma \geq \underline{\gamma^R}$ (or $\eta \leq \overline{\eta^R}$), a bank loan equilibrium with credit ratings is always more efficient than that without credit ratings.*

Proof. See Appendix. ■

Proposition 4 tells us that, so long as parameter values allow the three-tier credit rating system to exist, it is always more efficient than a banking system without credit ratings. Credit ratings reduce the social cost by giving some of the defaulting borrowers–those with the rating A–a second chance rather than immediately excluding them from borrowing. By doing so, credit ratings create two tiers of punishment: downgrading from A to B and downgrading from B to C. To discourage borrowers from strategic default, the costs of being downgraded in both cases need to be greater than the amount of loan repayment. This requires a minimum aggregate gap between the value of rating A and that of rating C, which can only be guaranteed with a more severe punishment imposed on defaulting borrowers with B rating–a higher cutoff value of γ or a lower cutoff value of η compared with the cutoff values in a banking system without credit ratings. Higher γ or lower η has two conflicting effects on welfare. On the one hand, it makes it possible to exempt defaulting borrowers with A rating from credit exclusion, which reduces social costs; on the other hand, it shuts defaulting borrowers with B ratings out of the capital market more often and for a longer period, which increases social costs. As we show below, the optimal credit rating system in the general case balances the trade-off between these two opposing effects.

5.2 The General Rating System

In this subsection, we extend the analysis to a general rating system that consists of N different ratings, indexed as $1, 2, ...N-1$, N, from the best to the worst. If an agent borrows and repays the loan, then his rating is upgraded one level above except for agents with rating 1, who keep the original rating 1. If an agent borrows and defaults, then his rating is downgraded one level except for agents with ratings $N-1$, who is downgraded to N with probability γ. An agent with rating N is excluded from the capital market in the current period but has a chance to be upgraded to rating $N-1$ next period, which happens with probability η; with probability $1-\eta$, he remains the original rating N next period. We use superscripts $G(k)$ $(k = 1, 2, ...N-1,\ N)$ to differentiate agents with ratings $1, 2, ...N-1,\ N$ respectively.

Agents with rating 1 have the following value functions once the capital and productivity shocks are realized:

$$\begin{aligned}
V_{1H}^{G(1)} &= \frac{1}{1+r}\{X_H + V^{G(1)}\}, \\
V_{1L}^{G(1)} &= \frac{1}{1+r}\{R_d + V^{G(1)}\}, \\
V_{0H}^{G(1)} &= \frac{1}{1+r}\{\pi(X - R_l + V^{G(1)}) + (1-\pi)V^{G(2)}]\}, \\
V_{0L}^{G(1)} &= \frac{V^{G(1)}}{1+r},
\end{aligned}$$

where

$$V^{G(1)} \equiv cpV_{1H}^{G(1)} + c(1-p)V_{1L}^{G(1)} + (1-c)pV_{0H}^{G(1)} + (1-c)(1-p)V_{0L}^{G(1)}$$

is the unconditional expected lifetime value at the beginning of a period.

Agents with rating k $(k = 2, 3, ...N-2)$ have the following value functions once the capital and productivity shocks are realized:

$$\begin{aligned}
V_{1H}^{G(k)} &= \frac{1}{1+r}\{X_H + V^{G(k)}\}, \\
V_{1L}^{G(k)} &= \frac{1}{1+r}\{R_d + V^{G(k)}\}, \\
V_{0H}^{G(k)} &= \frac{1}{1+r}\{\pi(X - R_l + V^{G(k-1)}) + (1-\pi)V^{G(k+1)}\}, \\
V_{0L}^{G(k)} &= \frac{V^{G(k)}}{1+r},
\end{aligned}$$

where

$$V^{G(k)} \equiv cpV_{1H}^{G(k)} + c(1-p)V_{1L}^{G(k)} + (1-c)pV_{0H}^{G(k)} + (1-c)(1-p)V_{0L}^{G(k)}$$

is the unconditional expected lifetime value at the beginning of a period.

Agents with rating $N-1$ have the following value functions once the capital and productivity shocks are realized:

$$\begin{aligned}
V_{1H}^{G(N-1)} &= \frac{1}{1+r}\{X_H + V^{G(N-1)}\}, \\
V_{1L}^{G(N-1)} &= \frac{1}{1+r}\{R_d + V^{G(N-1)}\}, \\
V_{0H}^{G(N-1)} &= \frac{1}{1+r}\{\pi(X - R_l + V^{G(N-2)}) + (1-\pi)[(1-\gamma)V^{G(N-1)} + \gamma V^{G(N)}]\}, \\
V_{0L}^{G(N-1)} &= \frac{V^{G(N-1)}}{1+r},
\end{aligned}$$

where

$$V^{G(N-1)} \equiv cpV_{1H}^{G(N-1)} + c(1-p)V_{1L}^{G(N-1)} + (1-c)pV_{0H}^{G(N-1)} + (1-c)(1-p)V_{0L}^{G(N-1)}$$

is the unconditional expected lifetime value at the beginning of a period.

Finally, agents with rating N have the following unconditional expected lifetime value:

$$V^{G(N)} = \frac{1}{1+r}\{cpX_H + c(1-p)X_L + \eta V^{G(N-1)} + (1-\eta)V^{G(N)}\}.$$

The value functions are subject to following constraints:

1). Depositors are willing to deposit their capital in banks:

$$R_d \geq X_L;$$

2a). Borrowers with rating 1 are willing to repay bank loans:

$$R_l \leq V^{G(1)} - V^{G(2)};$$

2b). Borrowers with rating k ($k = 2, 3, ...N-2$) are willing to repay bank loans:

$$R_l \leq V^{G(k-1)} - V^{G(k+1)};$$

2c). Borrowers with rating $N-1$ are willing to repay bank loans:

$$R_l \leq V^{G(N-2)} - [\gamma V^{G(N)} + (1-\gamma)V^{G(N-1)}];$$

3). Banks break even:

$$\pi R_l \geq R_d;$$

4). A constant steady state population distribution:

$$\begin{aligned} \alpha^{G(k-1)}(1-c)p(1-\pi) &= \alpha^{G(k)}(1-c)p\pi \qquad k=1,2,...N-1, \\ \alpha^{G(N-1)}(1-c)p(1-\pi)\gamma &= (1-\textstyle\sum_{k=1}^{N-1}\alpha^{G(k)})\eta, \end{aligned}$$

where $\alpha^{G(k)}$denotes the proportion of agents with rating k $(k=1,2,...N-1)$.

Same as before, we assume that competition drives the deposit rate and the loan rate to the minimum level; that is, $R_d = X_L$ and $\pi R_l = X_L$. It is trivial to see that the expected lifetime value decreases as an agent's rating deteriorates. As a matter of fact, the rating system creates a chain of incentive compatibility constrains that gives every borrower a carrot-and-stick choice: a rating upgrade for repayment or a rating downgrade for default. In equilibrium, all borrowers with high outputs choose the carrot. Proposition 5 characterizes the equilibrium solutions of the value functions, which can be represented with two series of recursive functions.

Proposition 5 *If an equilibrium with N ratings exists, the value functions are represented as:*

$V^{G(k)} = \frac{cpX_H + c(1-p)X_L + (1-c)p(X_H - X_L)}{r} - Y^{G(k)}$, *where $Y^{G(k)}$ follows the following recursive rules:*

1)for $k=1,2,...N-1,\ N,\ m_1 = 0,$ and $m_{k+1} = \frac{r+(1-c)p\pi\cdot m_k}{r+(1-c)p(1-\pi)+(1-c)p\pi m_k}$;
2a)for agents with rating N, we have $Y^{G(N)} = \frac{(1-c)p(X_H-X_L)}{(r+\eta)-\frac{\eta\gamma(1-c)p(1-\pi)}{r+\gamma(1-c)p(1-\pi)+(1-c)p\pi m_{N-1}}}$;
2b) for agents with rating $k=N-1$, we have $Y^{G(N-1)} = \frac{\gamma(1-c)p(1-\pi)Y^{G(N)}}{r+\gamma(1-c)p(1-\pi)+(1-c)p\pi m_{N-1}}$;

2c)for agents with rating $k = 1, 2, ...N-2$*, we have* $Y^{G(k)} = \frac{(1-c)p(1-\pi)Y^{G(K+1)}}{r+(1-c)p(1-\pi)+(1-c)p\pi m_k}$.

Proof. See Appendix. ■

In order for a steady state equilibrium to exist, the solutions need to satisfy all the constraints, among which the incentive compatibility constraints are the most critical to the prevention of strategic default. The following lemma simplifies the analysis and enables us to pin down the condition under which a steady state equilibrium exists.

Lemma 1 *For* $k = 1, 2, ...N-2$*, the incentive compatibility constraint of agents with rating* k *subsumes that of agents with rating* $k+1$*.*

Proof. See Appendix. ■

Lemma 1 essentially says that the only incentive compatibility constraint that matters is that of agents with the best rating. In other words, as an agent's rating drops, the cost of default increases at an accelerated speed. As a result, the incentive compatibility constraint of agents with the best rating determines whether an equilibrium exists.

Proposition 6 *In equilibrium, a rating system can only consist of a finite maximum number,* $\widehat{N}$*, of ratings, with* $\widehat{N}$ *determined by the incentive compatibility condition of agents with the best rating. Moreover,* $\widehat{N}$ *is increasing in* γ *and decreasing in* η*. If an equilibrium with* $\widehat{N}$ *ratings exists, then there also exist equilibria with* $2, 3, ...\widehat{N}-1$ *ratings, but the equilibrium with* $\widehat{N}$ *ratings is the most efficient.*

Proof. See Appendix. ■

Proposition 6 again highlights the opposing effects of credit exclusion on social welfare. A more severe punishment of defaulters with the lowest ratings is costly because their high productivity will be idled for a longer time. Nevertheless, it raises the amount of soft collateral of those agents with better ratings and thus allows for additional tiers of ratings, which in turn means that fewer defaulting borrowers need to be excluded from the capital market. The trade-off between the number of agents excluded from the capital market versus the average time-length of credit exclusion determines the optimal rating system.

5.3 The Optimal Rating System

Our analysis above shows that the allowed maximum number of ratings is increasing in the severity of the punishment imposed on defaulting borrowers with the worst rating: the probability of defaulting borrowers with the rating $N-1$ to be excluded from borrowing, γ, and the chance of those excluded agents to be absolved and allowed to borrow again, η. Since credit exclusion precedes forgiveness and absolution, the parameter γ plays a more important role than η.

Proposition 7 *In an equilibrium with credit ratings, social welfare only depends on the ratio of γ to η. Given the ratio γ/η, a greater value of γ allows a weakly more efficient equilibrium.*

Proof. See Appendix. ■

Based on Proposition 7, we can set γ equal to one and analyze the effect of η on social welfare. On the one hand, a lower η allows a greater number of ratings and fewer defaulting borrowers need to be excluded from the capital market; on the other hand, a lower lower η implies that it is more difficult for agents who are shut out of the credit market to come back. In the extreme case, when η goes to zero, almost every agent is prohibited from borrowing and we essentially retrogress to autarky, which is the most inefficient case. Therefore, there must exist an interior solution to η that delivers the optimal social welfare.

Proposition 8 *There exists an interior $\eta^* \in (0,1)$ that determines the optimal number of ratings and delivers the optimal social welfare.*

Proof. See Appendix. ■

A lower η makes it more difficult for an excluded borrower to get a fresh start; however, the more severe punishment enables the system to increase the number of ratings and give an average borrower more chances to repair his credit rating before he hits the worst rating and is excluded. In other words, there is a trade-off between how often versus how long an agent is excluded from the capital market. Since credit exclusion is the source of inefficiency, the optimal value of η minimizes the social cost by minimizing the steady state population of agents who are excluded from borrowing. Furthermore, the optimal value of η determines the optimal tiers of credit ratings in the equilibrium.

6 Differential Loan Rates

So far we have assumed that, if a borrower is not excluded from the credit market, then the interest rate he pays is the same regardless of his rating. This feature is different from other papers in the literature, such as Diamond (1989), Vercammen (1995), and Padilla and Pagano (2000), that use interest rate differentiation to incentivize borrowers. While these papers are based on unobservable ex ante heterogenous borrow qualities, our paper is based on the assumption that borrowers' are of the same quality; as a result, a borrower's rating does not convey any information about the repayment ability. This assumption allows us to zero in on the disciplinary function of credit ratings. We will show that, in our framework, interest rate differentiation alone cannot achieve the same disciplinary effect as credit exclusion does, but the combination of rate differentiation and credit exclusion can improve social welfare.

To prove that rate differentiation alone does not work, we only need to consider the three-tier rating system we solved in Section 5.1. When there is no credit exclusion, we can reduce the system to a two-tier one. We assume that borrowers with different ratings are charged with different interest rates: R_l^A and R_l^B for agents with ratings A and B respectively. To save space, we omit the value functions and list the modified incentive compatibility conditions as follows:

1). Depositors are willing to deposit their capital in banks:

$$R_d \geq X_L;$$

2). Borrowers with ratings A and B are willing to repay bank loans:

$$R_l^A \leq R_l^B \leq V^{RA} - V^{RB};$$

3). Banks break even:

$$\pi[\alpha^{RA} R_l^A + (1 - \alpha^{RA}) R_l^B] \geq R_d;$$

where α^{RA} and $(1 - \alpha^{RA})$ are the fractions of borrowers with ratings A and B respectively, because none of the agents is excluded from the market.

Same as before, we assume that competition drives the deposit rate and the loan rate to the minimum level; that is, $R_d = X_L$ and $\pi[\alpha^{RA} R_l^A + (1 - \alpha^{RA}) R_l^B] = X_L$.

Proposition 9 *Without credit exclusion, there does not exist a steady state equilibrium where borrowers with different ratings are charged with different interest rates.*

Proof. See Appendix. ■

The reason interest rate differentiation alone cannot support a steady state equilibrium is that the value functions are endogenized. When the interest rates are the same ($R_l^A = R_l^B$), the expected lifetime values are also the same ($V^{RA} = V^{RB}$). As we increase the difference between R_l^A and R_l^B, the difference between V^{RA} and V^{RB} also increases. However, the difference between the two expected lifetime values does not increase as fast as the difference between the two interest rates because future production shocks are independent of ratings. As a result, interest rate differentiation cannot satisfy the incentive compatibility conditions. To make the rating system work, credit exclusion is indispensable. This is analogous to the result in partial equilibrium dynamic contracting papers (e.g., Demarzo and Fishman (2007) and Biais et al. (2007)) that a possibility of termination or liquidation is necessary to deter moral hazard.

Next we consider the combination of rate differentiation and credit exclusion in the case with three ratings. Based on Proposition 7, we set γ equal to one and assume that η is between zero and one. In other words, if an agent with rating B defaults, he will be downgraded to rating C and excluded from borrowing next period; the exclusion is lifted with probability η starting from the period after the next. With differential rates, the incentive compatibility constraint of agents with rating B needs to be modified as:

$$R_l^B \leq V^{RA} - V^{RC}.$$

The following proposition compares the rating system with differential loan rates and the rating system with equal loan rates we characterized in Section 5.1.

Proposition 10 *Let $\gamma = 1$ and $\widehat{\eta^R}$ ($\overline{\eta^R}$) denote the maximum value of η such that a three-tier rating system with differential loan rates (equal loan rates) exists. We have $\widehat{\eta^R} > \overline{\eta^R}$; that is, when $\widehat{\eta^R} \geq \eta > \overline{\eta^R}$, there exists a three-tier rating system with differential loan rates but not a three-tier rating system with equal loan rates. Consequently, the rating system with differential loan rates is more efficient than that with equal loan rates.*

Proof. See Appendix. ■

In the case with equal loan rates, the maximum value of η is obtained when the incentive compatibility constraint of agents with rating A is binding. If we lower the loan rate for agents with rating A by a small amount, we can relax their incentive compatibility constraint without violating the incentive compatibility constraint of agents with rating B. As a result, we can allow agents with rating C to return to the credit market sooner. Since credit exclusion is the only source of inefficiency in the model, a greater chance of returning to the credit market improves social welfare.

Now that we get the basic intuition from the case with three ratings, we proceed to analyze the general case with N ratings. Specifically, in the general case we analyzed in Section 5.2, we assume that agents with ratings $1, 2, ...N-1$ need to pay loan rates R_l^1, R_l^2, $...R_l^{N-1}$ respectively. Agents with rating N will be excluded from borrowing for at least one period. For ease of exposition, we again omit the value functions and list the modified incentive compatibility conditions as follows:

1). Depositors are willing to deposit their capital in banks:

$$R_d \geq X_L;$$

2a). Borrowers with rating 1 are willing to repay bank loans:

$$R_l^1 \leq V^{G(1)} - V^{G(2)};$$

2b). Borrowers with rating k ($k = 2, 3, ...N-1$) are willing to repay bank loans:

$$R_l^k \leq V^{G(k-1)} - V^{G(k+1)};$$

3). Banks break even:

$$\pi \sum_{k=1}^{N-1} \alpha^{G(k)} R_l^k \geq R_d \sum_{k=1}^{N-1} \alpha^{G(k)};$$

4). A constant steady state population distribution:

$$\begin{aligned} \alpha^{G(k-1)}(1-c)p(1-\pi) &= \alpha^{G(k)}(1-c)p\pi \qquad k = 1, 2, ...N-1, \\ \alpha^{G(N-1)}(1-c)p(1-\pi) &= (1 - \textstyle\sum_{k=1}^{N-1} \alpha^{G(k)})\eta, \end{aligned}$$

where $\alpha^{G(k)}$denotes the proportion of agents with rating k ($k = 1, 2, ...N-1$).

As before, we assume that competition drives the deposit rate to the minimum level and banks' profits to zero; that is, $R_d = X_L$ and $\pi \sum_{k=1}^{N-1} \alpha^{G(k)} R_l^k = X_L \sum_{k=1}^{N-1} \alpha^{G(k)}$.

Lemma 1 in Section 5 shows that, in a rating system with equal loan rates, only the incentive compatibility constraint of agents with the best rating is binding. Intuitively, if we lower the loan rate of these agents and increase the loan rates of all other agents who do not have the best rating, we can effectively relax the constraint for the agents with the best rating without violating the constraints for all other agents. The following proposition formally characterizes this intuition.

Proposition 11 *For any given $\eta \in (0,1)$, if there exists a rating system that consists of N ratings with equal loan rates, there also exist rating systems that consist of at least N ratings with differential loan rates.*

Proof. See Appendix. ■

The previous proposition implies that rate differentiation can help tighten agents' incentives. In our model, agents with different ratings have the same production technology and thus the same repayment ability, so rate differentiation only serves as an incentive device, which is a desirable supplement of credit exclusion because it increases the value of soft collateral, and by doing so, it can increase the number of ratings and reduce the necessity to exclude agents from borrowing. However, the critical trade-off between the frequency and severity of punishing defaulting agents with the lowest rating still exists and dictates the optimal rating system, as shown by the next proposition.

Proposition 12 *Even with rate differentiation, the optimal rating system is coarse and the number of rating is decreasing in η; as a result, there exists an interior $\eta^* \in (0,1)$ that determines the optimal number of ratings and delivers the optimal social welfare.*

Proof. See Appendix. ■

The analysis above shows that credit exclusion is crucial in preventing strategic default. Without credit exclusion, rate differentiation alone cannot provide agents with incentives to repay their loans. For agents with good ratings, banks can use higher loan rates associated with downgrading to deter strategic default. However, the highest loan rate that banks can charge is capped by the output level when the project succeeds, which implies that agents with the lowest ratings must be punished with credit exclusion. Nevertheless, once credit exclusion is a possibility, rate differentiation can further fine-tune the incentive mechanism and reduce the population that is actually shut out of the credit market.

7 Conclusion

In this paper, we show that credit exclusion is a form of soft collateral that can be used to alleviate a borrower's incentive to default strategically. This soft collateral is very weak in a dispersed individual loan market because search frictions decrease the probability of meeting a lender and, in the meantime, increase the difficulty of sharing information about defaulting borrowers. Banks arise to improve efficiency in the sense that a centralized market facilitates credit access, makes it easier to identify a defaulting borrower, and thus boosts the value of the soft collateral. A credit rating system, especially one with differential rates, can further improve efficiency because it stratifies the soft collateral and reduces the necessity of credit exclusion to agents with the lowest ratings, i.e., those with insufficient soft collateral.

Turning to actual credit markets, our model is closest to the way individual borrowing takes place in the U.S. Individuals are rated based on their past financial behavior, including late payments, defaults, etc. However, while these events are typically recorded by the major credit bureaus, federal law mandates that these records be expunged after a certain period of time (currently seven years for most negative reports). By contrast, corporate borrowers have no such protection; also, unlike human borrowers, corporations that file for bankruptcy may not emerge at all, or may do so with their former creditors owning them. Sole proprietorships are likely to be an intermediate case, since their creditworthiness often relies on the owner-operator's personal credit records.

In addition to the basic outlines of credit reporting, our model is also consistent with the use of credit grades. In this regard, note that while FICO credit scores are close to continuous with integer scores ranging from 300 to 850, broad categories with sharp cut-offs are used in many sectors. For example, in the residential mortgage market, major institutions such as Fannie Mae and Freddie Mac consider scores below 640 to be subprime, which they will not finance; other lenders charge such borrowers higher rates if they lend to them at all. Although individual credit scores fluctuate up and down constantly, the overall distribution is pretty stable. Turning to the corporate market, coarse ratings are the standard, with the three major agencies (S&P, Moody's, and Fitch) using a restricted set of letter grades with some subcategories.

Looking beyond the U.S., most European countries also mandate a maximum retention time for data on individual loan defaults, though the specific interval differs from country to country (Rothemund and Gerhardt, 2011). Given that our paper links optimal exclusion/reinstatement policies to a number of credit market features and fundamentals, further analysis of these differences would be a fruitful test of our model.

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Appendix

Proof of Proposition 1: Plugging $V_{1H}^A, V_{1L}^A, V_{0H}^A$, and V_{0L}^A into the expression for V^A, we can easily get:

$$
\begin{aligned}
V^A &\equiv cpV_{1H}^A + c(1-p)V_{1L}^A + (1-c)pV_{0H}^A + (1-c)(1-p)V_{0L}^A \\
&= \frac{1}{1+r}\{cpX_H + c(1-p)X_L + V^A\}.
\end{aligned}
$$

The solutions are:

$$
\begin{aligned}
V^A &= \frac{cpX_H + c(1-p)X_L}{r}, \\
V_{0H}^A &= V_{0L}^A = \frac{cpX_H + c(1-p)X_L}{(1+r)r}, \\
V_{1L}^A &= \frac{cpX_H + c(1-p)X_L}{(1+r)r} + \frac{X_L}{1+r}, \\
V_{1H}^A &= \frac{cpX_H + (1-p)X_L}{(1+r)r} + \frac{X_H}{1+r}.
\end{aligned}
$$

Proof of Proposition 2: Plugging $\pi R = X_L, V_{1H}^I, V_{1L}^I, V_{0H}^I$, and V_{0L}^I into the expression for V^I, we have:

$$
\begin{aligned}
[r + c(1-c)p(1-p)(1-\pi)\gamma]V^I &= cpX_H + c(1-p)X_L + c(1-c)p(1-p)(X_H - X_L) \\
&\quad + c(1-c)p(1-p)(1-\pi)\gamma V^{I(e)}.
\end{aligned}
$$

In combination with the equation for $V^{I(e)}$, we can get:

$$
\begin{aligned}
V^I &= \frac{cpX_H + c(1-p)X_L}{r} + \frac{(r+\eta)[c(1-c)p(1-p)\pi](X_H - X_L)}{r[r+\eta+c(1-c)p(1-p)(1-\pi)\gamma]}, \\
V^{I(e)} &= \frac{cpX_H + c(1-p)X_L}{r} + \frac{\eta[c(1-c)p(1-p)\pi](X_H - X_L)}{r[r+\eta+c(1-c)p(1-p)(1-\pi)\gamma]}.
\end{aligned}
$$

The borrower's incentive compatibility condition is:

$$\begin{aligned} X_L/\pi &\leq \gamma(V^I - V^{I(e)}) \\ &= \frac{[c(1-c)p(1-p)\pi](X_H - X_L)}{\frac{r+\eta}{\gamma} + c(1-c)p(1-p)(1-\pi)}. \end{aligned}$$

The incentive compatibility condition is satisfied if $\frac{r+\eta}{\gamma}$ is small enough; that is, given all other variables, the existence of a private loan equilibrium requires a minimum of γ denoted by $\underline{\gamma^I}$, or a maximum value of η denoted by $\overline{\eta^I}$.

Social welfare in this case is equal to the weighted average of the expected lifetime value, $W^I = \alpha^I V^I + (1-\alpha^I)V^{I(e)}$. Plugging in the steady state population distribution, $\alpha^I = \frac{1}{1+c(1-c)p(1-p)(1-\pi)\gamma/\eta}$, we have:

$$W^I = \frac{cpX_H + c(1-p)X_L}{r} + \frac{c(1-c)p(1-p)(X_H - X_L)}{r[1 + c(1-c)p(1-p)(1-\pi)\gamma/\eta]}.$$

Proof of Proposition 3: We assume the loan rate is set at the minimum; that is, $R_d = X_L$ and $\pi R_l = X_L$. Similar to the proof of Proposition 2, we can get:

$$\begin{aligned} V^B &= \frac{cpX_H + c(1-p)X_L}{r} + \frac{(r+\eta)(1-c)p\pi(X_H - X_L)}{r[r+\eta+(1-c)p(1-\pi)\gamma]} \\ V^{B(e)} &= \frac{cpX_H + c(1-p)X_L}{r+\eta} + \frac{\eta(1-c)p\pi(X_H - X_L)}{r[r+\eta+(1-c)p(1-\pi)\gamma]}. \end{aligned}$$

The borrower's incentive compatibility condition can be simplified as:

$$\frac{X_L}{\pi} \leq \frac{(1-c)p(X_H - X_L)}{\frac{r+\eta}{\gamma} + (1-c)p(1-\pi)}.$$

The incentive compatibility condition is satisfied if $\frac{r+\eta}{\gamma}$ is small enough. It is trivial to see that the required minimum value of $\frac{r+\eta}{\gamma}$ is greater than that for the existence of a private loan equilibrium, which implies relaxed cutoff values for γ and η: $\underline{\gamma^I} > \underline{\gamma^B}$; $\overline{\eta^I} < \overline{\eta^B}$.

Social welfare in the bank loan equilibrium is equal to the weighted average of the expected lifetime value, $W^B = \alpha^B V^B + (1-\alpha^B)V^{B(e)}$. Plugging in the steady state population distribution, $\alpha^B = \frac{1}{1+(1-c)p(1-\pi)\gamma/\eta}$,we have:

$$W^B = \frac{cpX_H + c(1-p)X_L}{r} + \frac{(1-c)p(X_H - X_L)}{r[1+(1-c)p(1-\pi)\gamma/\eta]}.$$

It is trivial to see that a bank loan equilibrium dominates a private loan equilibrium.

Proof of Proposition 4: We can reduce the equilibrium to the following three equations:

$$\begin{aligned}
[r+(1-c)p(1-\pi)]V^{RA} &= cpX_H + c(1-p)X_L + (1-c)p(X_H - X_L) \\
&\quad +(1-c)p(1-\pi)V^{RB}, \\
[r+(1-c)p\pi + (1-c)p(1-\pi)\gamma]V^{RB} &= cpX_H + c(1-p)X_L + (1-c)p(X_H - X_L) \\
&\quad +(1-c)p\pi V^{RA} + (1-c)p(1-\pi)\gamma V^{RC}, \\
[r+\eta]V^{RC} &= cpX_H + c(1-p)X_L + \eta V^{RB}.
\end{aligned}$$

The solutions are:

$$\begin{aligned}
V^{RA} &= \frac{cpX_H + c(1-p)X_L}{r} + \frac{(1-c)p(X_H - X_L)}{r\{1+\frac{[r+(1-c)p(1-\pi)](1-c)p(1-\pi)\gamma}{(r+\eta)[r+(1-c)p]}\}} \\
&\quad + \frac{(1-c)p(X_H - X_L)}{\frac{(r+\eta)[r+(1-c)p]}{(1-c)p(1-\pi)\gamma}\{1+\frac{[r+(1-c)p(1-\pi)](1-c)p(1-\pi)\gamma}{(r+\eta)[r+(1-c)p]}\}}, \\
V^{RB} &= \frac{cpX_H + c(1-p)X_L}{r} + \frac{(1-c)p(X_H - X_L)}{r\{1+\frac{[r+(1-c)p(1-\pi)](1-c)p(1-\pi)\gamma}{(r+\eta)[r+(1-c)p]}\}}, \\
V^{RC} &= \frac{cpX_H + c(1-p)X_L}{r} + \frac{\eta(1-c)p(X_H - X_L)}{r(r+\eta)\{1+\frac{[r+(1-c)p(1-\pi)](1-c)p(1-\pi)\gamma}{(r+\eta)[r+(1-c)p]}\}}.
\end{aligned}$$

Because $V^{RC} < V^{RB}$, the incentive compatibility constraint of agents with rating A implies that of agents with rating B. Hence the existence of a steady state equilibrium requires:

$$
\begin{aligned}
\frac{X_L}{\pi} &\leq V^{RA} - V^{RB} \\
&= \frac{(1-c)p(X_H - X_L)}{\frac{[r+(1-c)p](r+\eta)}{(1-c)p(1-\pi)\gamma} + [r+(1-c)p(1-\pi)]}.
\end{aligned}
$$

Since $r > 0$ and $\frac{r+(1-c)p}{(1-c)p(1-\pi)} > 1$, the required value of $\frac{(r+\eta)}{\gamma}$ for the existence of a steady state equilibrium is smaller than that for the existence of a bank loan equilibrium without credit rating; that is, $\underline{\gamma^R} > \underline{\gamma^B}$; $\overline{\eta^R} < \overline{\eta^B}$.

The steady state equilibrium population distribution is as follows:

$$
\begin{aligned}
\alpha^{RA} &= \frac{\pi}{1+(1-c)p(1-\pi)^2\gamma/\eta}, \\
\alpha^{RB} &= \frac{1-\pi}{1+(1-c)p(1-\pi)^2\gamma/\eta}.
\end{aligned}
$$

Social welfare is equal to:

$$
\begin{aligned}
W^R &= \alpha^{RA}V^{RA} + \alpha^{RB}V^{RB} + (1-\alpha^{RA}-\alpha^{RB})V^{RC} \\
&= \frac{cpX_H + c(1-p)X_L}{r} + \frac{(1-c)p(X_H - X_L)}{r[1+(1-c)p(1-\pi)^2\gamma/\eta]}.
\end{aligned}
$$

Compared with social welfare in a bank loan equilibrium without credit rating (W^B), it is trivial to see that credit rating improves efficiency so long as it exists in equilibrium.

Proof of Proposition 5: For $k = 1, 2, ...N-1,\ N$, defining

$$
Y^{G(k)} = \frac{cpX_H + c(1-p)X_L + (1-c)p(X_H - X_L)}{r} - V^{G(k)},
$$

we can transform the equations of value functions into the following expressions: for $k = 1$,

$$
[r+(1-c)p(1-\pi)]Y^{G(1)} = (1-c)p(1-\pi)Y^{G(2)};
$$

for $k = 2, 3, ...N-2$,

$$[(r + (1-c)p\pi + (1-c)p(1-\pi)]Y^{G(k)} = (1-c)p\pi Y^{G(k-1)} + (1-c)p(1-\pi)Y^{G(k+1)};$$

for $k = N-1$,

$$[(r + (1-c)p\pi + (1-c)p(1-\pi)\gamma]Y^{G(N-1)} = (1-c)p\pi Y^{G(N-2)} + (1-c)p(1-\pi)\gamma Y^{G(N)};$$

and for $k = N$,

$$(r+\eta)Y^{G(N)} = \eta Y^{G(N-1)} + (1-c)p(X_H - X_L).$$

We have

$$Y^{G(1)} = \frac{(1-c)p(1-\pi)Y^{G(2)}}{r + (1-c)p(1-\pi)} \text{ and}$$

$$Y^{G(k)} = \frac{(1-c)p(1-\pi)Y^{G(k+1)}}{r + (1-c)p(1-\pi) + (1-c)p\pi m_k},$$

which implies

$$\begin{aligned} 1 - m_{k+1} &= \frac{(1-c)p(1-\pi)}{r + (1-c)p(1-\pi) + (1-c)p\pi m_k}, \text{ or} \\ m_{k+1} &= \frac{r + (1-c)p\pi m_k}{r + (1-c)p(1-\pi) + (1-c)p\pi m_k}. \end{aligned}$$

For agents with rating $N-1$,we have

$$[(r + (1-c)p\pi + (1-c)p(1-\pi)\gamma]Y^{G(N-1)} = (1-c)p\pi Y^{G(N-2)} + (1-c)p(1-\pi)\gamma Y^{G(N)},$$

or

$$\begin{aligned} & [(r + (1-c)p\pi + (1-c)p(1-\pi)\gamma]Y^{G(N-1)} \\ = \; & \frac{(1-c)^2 p^2 \pi(1-\pi)Y^{G(N-1)}}{r + (1-c)p(1-\pi) + (1-c)p\pi m_{N-2}} + (1-c)p(1-\pi)\gamma Y^{G(N)}, \end{aligned}$$

which gives us

$$Y^{G(N-1)} = \frac{(1-c)p(1-\pi)\gamma Y^{G(N)}}{r + (1-c)p(1-\pi)\gamma + (1-c)p\pi m_{N-1}}.$$

Essentially we have a series of difference equations, which can be solved using induction. The solutions are:

1) for $k = 1, 2, ...N-1$, N, $m_1 = 0$, and $m_{k+1} = \frac{r+(1-c)p\pi \cdot m_k}{r+(1-c)p(1-\pi)+(1-c)p\pi m_k}$;

2a)for agents with rating N, we have $Y^{G(N)} = \frac{(1-c)p(X_H - X_L)}{(r+\eta) - \frac{\eta\gamma(1-c)p(1-\pi)}{r+\gamma(1-c)p(1-\pi)+(1-c)p\pi m_{N-1}}}$;

2b) for agents with rating $k = N-1$, we have $Y^{G(N-1)} = \frac{(1-c)p(1-\pi)\gamma Y^{G(N)}}{r+(1-c)p(1-\pi)\gamma+(1-c)p\pi m_{N-1}}$;

2c)for agents with rating $k = 1, 2, ...N-2$, we have $Y^{G(k)} = \frac{(1-c)p(1-\pi)Y^{G(k+1)}}{r+(1-c)p(1-\pi)+(1-c)p\pi m_k}$.

Proof of Lemma1: We can rewrite the incentive compatibility conditions as follows:

2a). for borrowers with rating 1:

$$R_l \leq Y^{G(2)} - Y^{G(1)};$$

2b). for borrowers with rating k ($k = 2, 3, ...N-2$):

$$R_l \leq Y^{G(k+1)} - Y^{G(k-1)};$$

2c). for borrowers with rating $N-1$:

$$R_l \leq \gamma Y^{G(N)} + (1-\gamma) Y^{G(N-1)} - Y^{G(N-2)}.$$

We first show that the incentive compatibility condition of borrowers with rating $k+1$ subsumes that of borrowers with rating k for $k = 2, 3...N-3$:

$$\left[Y^{G(k+2)} - Y^{G(k)}\right] - \left[Y^{G(k+1)} - Y^{G(k-1)}\right]$$

$$
\begin{aligned}
&= [Y^{G(k+2)} - Y^{G(k+1)}] - [Y^{G(k)} - Y^{G(k-1)}] \\
&= \frac{[r + (1-c)p\pi m_{k+1}]Y^{G(k+2)}}{r + (1-c)p(1-\pi) + (1-c)p\pi m_{k+1}} - \frac{[r + (1-c)p\pi m_{k-1}]Y^{G(k)}}{r + (1-c)p(1-\pi) + (1-c)p\pi m_{k-1}} \\
&> 0;
\end{aligned}
$$

the last step results from both $Y^{G(k)}$ and m_k being positive and increasing in k.

It is trivial to see that the incentive compatibility condition of borrowers with rating 2 implies that of borrowers with rating 1.As for borrowers with rating $N-1$,we can easily show that the incentive compatibility condition implies that of borrowers with rating $N-2$:

$$
\begin{aligned}
& [\gamma Y^{G(N)} + (1-\gamma)Y^{G(N-1)} - Y^{G(N-2)}] \\
> \; & Y^{G(N-1)} - Y^{G(N-2)} \\
> \; & Y^{G(N-1)} - Y^{G(N-3)}.
\end{aligned}
$$

Hence the only incentive compatibility condition that matters is that of borrowers with the best rating, 1.

Proof of Proposition 6: Since the expected lifetime payoff of agents with the best rating, $V^{G(1)}$, is capped by $\frac{cpX_H + c(1-p)X_L + (1-c)p(X_H - X_L)}{r}$, which would be achieved if they would never be excluded from borrowing, and the the expected lifetime payoff of agents excluded from borrowing, $V^{G(N)}$, is floored by $\frac{cpX_H + c(1-p)X_L}{r}$, which is the autarky value, the difference between these two values is finite and can only support a finite number of incentive compatibility conditions. Consequently, a rating system can only have a finite maximum number of ratings.

Following Lemma 1, the only incentive compatibility condition that matters is that of agents with the best rating, which can be expressed as:

$$
X_L/\pi \leq Y^{G(2)} - Y^{G(1)} = \frac{r}{(1-c)p(1-\pi)} Y^{G(1)}.
$$

In order to prove that the allowed maximum number of ratings, $\widehat{N}$, is increasing in γ and decreasing in η, we only need to show that $Y^{G(1)}$ is increasing in γ and decreasing in η. As Proposition 5 shows that, $Y^{G(1)}$ depends on the m_k's and $Y^{G(N-1)}$. Because m_k's do not depend on γ or η, it boils down to showing that $Y^{G(N-1)}$ is increasing in γ and decreasing in η, as shown below:

$$\begin{aligned}
Y^{G(N-1)} &= \frac{\gamma(1-c)p(1-\pi)Y^{G(N)}}{r+\gamma(1-c)p(1-\pi)+(1-c)p\pi\cdot m_{N-1}} \\
&= \frac{\gamma(1-c)p(1-\pi)}{r+\gamma(1-c)p(1-\pi)+(1-c)p\pi\cdot m_{N-1}} \frac{(1-c)p(X_H-X_L)}{r+\eta-\frac{\eta\gamma(1-c)p(1-\pi)}{r+\gamma(1-c)p(1-\pi)+(1-c)p\pi\cdot m_{N-1}}} \\
&= \frac{(1-c)^2p^2(1-\pi)(X_H-X_L)}{\frac{r+\eta}{\gamma}[r+(1-c)p\pi\cdot m_{N-1}]+r(1-c)p(1-\pi)}.
\end{aligned}$$

Next, we show that if an equilibrium with N ratings exists, then there also exists an equilibrium with $N-1$ ratings; hence, by induction, there exist equilibria with $2,3,...N-1$ ratings. Suffice it to show that $Y^{G(1)}$ in a system with N ratings is smaller than $Y^{G(1)}$ in a system with $N-1$ ratings. Compared with a system with N ratings, the value functions in a system with $N-1$ ratings depends on the exact same series of m_k's except that is truncated at m_{N-2} because $N-2$ is the rating next to the last rating, that is, $N-1$. Because $m_{N-2} < m_{N-1}$, it is trivial to see that $Y^{G(N-2)}$ in a system with N ratings is smaller than $Y^{G(N-2)}$ in a system with $N-1$ ratings. As a result, $Y^{G(1)}$ in a system with N ratings is also smaller than $Y^{G(1)}$ in a system with $N-1$ ratings.

Finally, we examine social welfare, which depends on the steady state distribution of agents with different ratings. Because we have $\alpha^{G(k)} = \frac{\pi}{1-\pi}\alpha^{G(k-1)}$, the fraction of agents with rating N is:

$$1-\sum_{k=1}^{N-1}\alpha^{G(k)} = 1-\alpha^{G(N-1)}\frac{1-(\frac{\pi}{1-\pi})^{N-1}}{1-\frac{\pi}{1-\pi}}.$$

Using the steady state equilibrium condition

$$\alpha^{G(N-1)}(1-c)p(1-\pi)\gamma = (1-\sum_{k=1}^{N-1}\alpha^{G(k)})\eta,$$

we can get

$$\alpha^{G(N-1)} = \frac{1}{\frac{1-(\frac{\pi}{1-\pi})^{N-1}}{1-\frac{\pi}{1-\pi}} + (1-c)p(1-\pi)\gamma/\eta}.$$

So the fraction of agents with rating N is equal to

$$1 - \sum_{k=1}^{N-1} \alpha^{G(k)} = \frac{(1-c)p(1-\pi)\gamma/\eta}{\frac{1-(\frac{\pi}{1-\pi})^{N-1}}{1-\frac{\pi}{1-\pi}} + (1-c)p(1-\pi)\gamma/\eta}.$$

Social welfare is the first best solution, $\frac{cpX_H+c(1-p)X_L+(1-c)p(X_H-X_L)}{r}$ (no agent is excluded from borrowing), minus the loss due to agents with rating N being excluded:

$$W^G = \frac{cpX_H + c(1-p)X_L + (1-c)p(X_H - X_L)}{r} - \frac{(1-\sum_{k=1}^{N-1}\alpha^{G(k)})(1-c)p(X_H - X_L)}{r}.$$

Since $1 - \sum_{k=1}^{N-1} \alpha^{G(k)}$ is decreasing in N given the primitive parameters, the rating system with the maximum number of ratings, $\widehat{N}$, is the most efficient.

Proof of Proposition 7: Proposition 6 shows that social welfare is only affect by γ/η. On the other hand, the incentive compatibility condition only depends on on $Y^{G(1)}$, which can be obtained through $Y^{G(N-1)}$ and m_k's. $Y^{G(N-1)}$ is related to γ and η through the term $\frac{r+\eta}{\gamma}$, while m_k's are not affected by either γ or η. Hence if we maintain the value of γ/η and increase γ, we can achieve the same social welfare and relax the incentive compatibility constraint; in addition, if the increase in γ allows us to add one more tier of rating, social welfare can be improved.

Proof of Proposition 8: Proposition 6 shows that there exists a finite maximum number of ratings, $\widehat{N}$, and $\widehat{N}$ is decreasing in η. As η goes to zero, the number of ratings converges to its maximum; however, social welfare retrogresses to the case of autarky because almost every agent is excluded from borrowing. So an optimal rating system is not the one that pushes the number of ratings to the maximum.

Proof of Proposition 9: We can reduce the equilibrium to the following two equations:

$$\begin{aligned}
[r+(1-c)p(1-\pi)]V^{RA} &= cpX_H + c(1-p)X_L + (1-c)p(X_H - \pi R_l^A) \\
&\quad + (1-c)p(1-\pi)V^{RB}, \\
[r+(1-c)p\pi + (1-c)p(1-\pi)]V^{RB} &= cpX_H + c(1-p)X_L + (1-c)p(X_H - \pi R_l^B) \\
&\quad + (1-c)p\pi V^{RA},
\end{aligned}$$

which gives us:

$$(1-c)p(\pi R_l^B - \pi R_l^A) = [r+(1-c)p](V^{RA} - V^{RB}).$$

It can be seen immediately that it contradicts the incentive compatibility condition $R_l^A \leq R_l^B \leq V^{RA} - V^{RB}$.

Proof of Proposition 10: With differential rates. We have:

$$\begin{aligned}
\beta V^{RA} &= cpX_H + c(1-p)X_L + (1-c)p\pi(X - R_l^A) + (1-c)p(1-\pi)(V^{RB} - V^{RA}) \\
\beta V^{RB} &= cpX_H + c(1-p)X_L + (1-c)p\pi(X - R_l^B) + (1-c)p\pi(V^{RA} - V^{RB}) \\
&\quad + (1-c)p(1-\pi)(V^{RC} - V^{RB}) \\
(r+\eta)\, V^{RC} &= cpX_H + c(1-p)X_L + \eta V^{RB}.
\end{aligned}$$

If we impose the condition $R_l^A = R_l^B = \frac{X_L}{\pi}$, we go back to the case with equal loan rates, whose value functions we denote by $(\overline{V}^{RA}, \overline{V}^{RB}, \overline{V}^{RC})$. Solving the equations above, we get:

$$\begin{aligned}
V^{RB} - \overline{V}^{RB} &= -\frac{(1-c)p\pi^2(R_l^B - R_l^A)}{r+(1-c)p + \frac{[r+(1-c)p(1-\pi)](1-c)p(1-\pi)}{(r+\eta)}} \\
V^{RC} - \overline{V}^{RC} &= \frac{\eta}{r+\eta}(V^{RB} - \overline{V}^{RB}) \\
&= -\frac{\eta(1-c)p\pi^2(R_l^B - R_l^A)}{(r+\eta)[r+(1-c)p] + [r+(1-c)p(1-\pi)](1-c)p(1-\pi)} \\
V^{RA} - V^{RB} &= \frac{(1-c)p\pi(R_l^B - R_l^A)}{r+(1-c)p} + \frac{(1-c)p(1-\pi)[\beta V^{RB} - cpX_H + c(1-p)X_L]}{[r+(1-c)p](r+\eta)}
\end{aligned}$$

$$
\begin{aligned}
&= (\overline{V}^{RA} - \overline{V}^{RB}) + \frac{(1-c)p\pi(R_l^B - R_l^A)}{r + (1-c)p} + \frac{(1-c)p(1-\pi)r(V^{RB} - \overline{V}^{RB})}{[r+(1-c)p](r+\eta)} \\
&= (\overline{V}^{RA} - \overline{V}^{RB}) + \frac{[(r+\eta) + (1-c)p(1-\pi)^2](1-c)p\pi(R_l^B - R_l^A)}{(r+\eta)[r+(1-c)p] + [r+(1-c)p(1-\pi)](1-c)p(1-\pi)} \\
V^{RA} - \overline{V}^{RA} &= \frac{[(r+\eta) + (1-c)p(1-\pi)](1-c)p\pi(1-\pi)(R_l^B - R_l^A)}{(r+\eta)[r+(1-c)p] + [r+(1-c)p(1-\pi)](1-c)p(1-\pi)}.
\end{aligned}
$$

Hence we have $V^{RB} < \overline{V}^{RB}$, $V^{RC} < \overline{V}^{RC}$, but $V^{RA} < \overline{V}^{RA}$.

Incentive compatibility conditions require:

$$
\begin{aligned}
R_l^A &\leq V^{RA} - V^{RB} \\
R_l^B &\leq V^{RA} - V^{RC}.
\end{aligned}
$$

And banks' zero profit condition requires:

$$
\alpha^{RA} R_l^A + \alpha^{RB} R_l^B = (\alpha^{RA} + \alpha^{RB}) \frac{X_L}{\pi}, \text{ or } \pi R_l^A + (1-\pi) R_l^B = \frac{X_L}{\pi}.
$$

Let $R_l^A = \frac{X_L}{\pi} - \Delta$, then $R_l^B = \frac{X_L}{\pi} + \frac{\pi}{1-\pi}\Delta$. The incentive compatibility conditions can be rewritten as:

$$
\begin{aligned}
\Delta &\geq \frac{\frac{X_L}{\pi} - (\overline{V}^{RA} - \overline{V}^{RB})}{\frac{[r+(1-c)p][(r+\eta)+(1-c)p(1-\pi)] + (r+\eta)(1-c)p\frac{\pi}{1-\pi}}{(r+\eta)[r+(1-c)p] + [r+(1-c)p(1-\pi)](1-c)p(1-\pi)}} \\
\Delta &\leq \frac{(\overline{V}^{RA} - \overline{V}^{RC}) - \frac{X_L}{\pi}}{\frac{r[(r+\eta)+(1-c)p]\frac{\pi}{1-\pi}}{(r+\eta)[r+(1-c)p] + [r+(1-c)p(1-\pi)](1-c)p(1-\pi)}}.
\end{aligned}
$$

Since we require $\Delta \geq 0$, the maximum value of η ($\overline{\eta^R}$) that allows a rating system with equal loan rates is obtained when $\frac{X_L}{\pi} = (\overline{V}^{RA} - \overline{V}^{RB})$. A system with differential loan rates allows η to be greater than $\overline{\eta^R}$, in which case we have $\frac{X_L}{\pi} - (\overline{V}^{RA} - \overline{V}^{RB}) > 0$. That is, $\overline{\eta^R}$ can serve as a lower bound for the maximum value of η allowed in a rating system with differential loan rates. On the other hand, the value of η has an upper bound because it is capped by the incentive compatibility constraint of agents with rating B, which makes

$(\overline{V}^{RA} - \overline{V}^{RC}) - \frac{X_L}{\pi} = 0$. Therefore, in the bounded region, there must exist an interior value, $\widehat{\eta^R} > \overline{\eta^R}$, that satisfy both incentive compatible constraints. This maximum value is also the most efficient one because the social welfare is increasing in η.

Proof of Proposition 11: Suppose there exists a rating system with N ratings and equal loan rates, whose solutions $(Y^{G(k)}, V^{G(k)})$ are given by Proposition 5. With differential loan rates, for $k = 1, 2, ...N-1,\ N$, we use $\widehat{V}^{G(k)}$ to denote the value functions.

We first increase all agents' loan rate by a same amount ε, which is infinitesimally small. Proposition 5 shows that all $Y^{G(k)}$'s are proportional, which means that all $Y^{G(k)}$'s will increase by a same proportion, which we denote by ρ. We use Δ to denote changes in the value functions, and we have

$$\begin{aligned} & \Delta[V^{G(1)} - V^{G(2)}] \\ = & \Delta[Y^{G(2)} - Y^{G(1)}] \\ = & \rho[Y^{G(2)} - Y^{G(1)}] \\ > & 0, \end{aligned}$$

which means that the incentive compatibility constraint for agents with the best rating $(k = 1)$ has been relaxed.

Next we reduce the loan rate of agents with the best rating such that banks earn zero profit; that is, $R_l^1 = R_l - \widehat{\varepsilon}$ and $R_l^k = R_l + \varepsilon$ $(k = 2, 3, ...N-1)$ such that $\varepsilon \sum_{k=2}^{N-1} \alpha^{G(k)} - \alpha^{G(1)} \widehat{\varepsilon} = 0$. In this case, we have differential rates, and we have

$$[r + (1-c)p](\widehat{V}^{G(1)} - \widehat{V}^{G(2)}) = (1-c)p\pi\widehat{\varepsilon} + (1-c)p(1-\pi)[(\widehat{V}^{G(2)} - \widehat{V}^{G(3)}).$$

Since $(\widehat{V}^{G(1)} - \widehat{V}^{G(2)})$ is increasing in $\widehat{\varepsilon}$, the incentive compatibility constraint is further relaxed compared to the case when all agents' loan rates have been increased by ε. In addition, when ε is small, all other agents' incentive compatibility constrains will still hold.

As a result, we have constructed a rating system with N ratings and differential loan rates.

Proof of Proposition 12: Suppose that the maximum number of ratings is N for a given value of η. It is trivial to show that N is finite. First, same as in the case with equal loan rates, the value of agents with the best rating is capped by no exclusion and the value of agents with the worst rating is floored at the autarky level. Consequently, the incentive compatibility conditions imply that the rating system only allows a finite number of ratings with loan rates greater than $R_l = \frac{X_L}{\pi}$. Second, the zero profit condition for banks implies that there can only be a finite number of ratings with loan rates smaller than $R_l = \frac{X_L}{\pi}$; otherwise the average loan rate could only be less than $R_l = \frac{X_L}{\pi}$.

We next show that as η decreases, the maximum number of ratings increases. Suppose the maximum number of ratings allowed for η is $N(\eta)$ and the value functions are denoted by $\widehat{V}_\eta^{G(k)}$, $k = 1, 2, 3, ...N(\eta)$. Now let η decrease to η'. We construct a rating system with the loan rates being the same for agents with ratings $k = 1, 2, 3, ...N(\eta) - 2$, and the loan rate for agents with rating $N(\eta) - 1$ is adjusted to $\widehat{R}_l^{N(\eta)-1}$ such that all agents with ratings $k = 1, 2, 3, ...N(\eta) - 1$ have the same expected lifetime value as in the case: $\widehat{V}_{\eta'}^{G(k)} = \widehat{V}_\eta^{G(k)}$. In other words, we want to have

$$\begin{aligned} r\widehat{V}_{\eta'}^{G(N(\eta)-1)} &= cpX_H + c(1-p)X_L + (1-c)p\pi(X - \widehat{R}_l^{N(\eta)-1}) \\ &\quad + (1-c)p\pi(\widehat{V}_{\eta'}^{G(N(\eta)-2)} - \widehat{V}_{\eta'}^{G(N(\eta)-1)}) + (1-c)p(1-\pi)(\widehat{V}_{\eta'}^{G(N(\eta))} - \widehat{V}_{\eta'}^{G(N(\eta)-1)}) \\ &= r\widehat{V}_\eta^{G(N(\eta)-1)}. \end{aligned}$$

This requires that

$$\widehat{R}_l^{N(\eta)-1} - R_l^{N(\eta)-1} = \frac{1-\pi}{\pi}(\widehat{V}_{\eta'}^{G(N(\eta))} - \widehat{V}_\eta^{G(N(\eta)-1)}).$$

On the other hand, if $r\widehat{V}_{\eta'}^{G(N(\eta)-1)} = r\widehat{V}_{\eta}^{G(N(\eta)-1)}$ we have

$$\begin{aligned}(r+\eta')\widehat{V}_{\eta'}^{G(N(\eta))} &= cpX_H + c(1-p)X_L + \eta'\widehat{V}_{\eta'}^{G(N(\eta)-1)} \\ &= cpX_H + c(1-p)X_L + \eta'\widehat{V}_{\eta}^{G(N(\eta)-1)} \\ &= (r+\eta)\widehat{V}_{\eta}^{G(N(\eta))} + (\eta'-\eta)\widehat{V}_{\eta}^{G(N(\eta)-1)} \\ &< (r+\eta)\widehat{V}_{\eta}^{G(N(\eta))} + (\eta'-\eta)\widehat{V}_{\eta}^{G(N(\eta))} \\ &= (r+\eta')\widehat{V}_{\eta}^{G(N(\eta))}.\end{aligned}$$

Therefore, we have

$$\widehat{R}_l^{N(\eta)-1} < R_l^{N(\eta)-1} \leq (\widehat{V}_{\eta}^{G(N(\eta)-2)} - \widehat{V}_{\eta}^{G(N(\eta))}) < (\widehat{V}_{\eta'}^{G(N(\eta)-2)} - \widehat{V}_{\eta'}^{G(N(\eta))}),$$

which means $\widehat{R}_l^{N(\eta)-1}$ is indeed incentive compatible. Thus we have shown that the maximum number of ratings is weakly decreasing in η.

As η goes to zero, the number of ratings converges to its maximum; however, social welfare retrogresses to the case of autarky because almost every agent is excluded from borrowing. So an optimal rating system is not the one that pushes the number of ratings to the maximum.