Asymmetric Information and Security Design under Knightian Uncertainty*

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Abstract

We study optimal security design by an informed issuer when the investor faces Knightian uncertainty about the distribution of cash flows and demands robustness: she evaluates each security by the worst-case distribution at which she could justify it being offered by the issuer. First, we show that both standard outside equity and standard risky debt arise as equilibrium securities. Thus, the model provides a common foundation for two most widespread financial contracts based on one simple market imperfection, information asymmetry. Second, we show that the equilibrium security differs depending on the degree of uncertainty and on whether private information concerns assets in place or the new project. If private information concerns the new project and uncertainty is sufficiently high, standard equity arises as the unique equilibrium security. When uncertainty is sufficiently small, the equilibrium typically features risky debt. In the intermediate case, both risky debt and standard equity arise in equilibrium. In contrast, if private information concerns assets in place, standard equity is never issued in equilibrium, irrespective of the level of uncertainty, and the equilibrium security is (usually) risky debt.

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1 Introduction

Consider a firm with insufficient internal capital that needs to raise the extra from the investor to finance an investment project. The firm's owner knows the distribution of the project's cash flows, while the investor does not. The firm proposes a security to the investor, which the investor uses to infer information about the distribution of the project's cash flows. How will this security look like? This is the classic question of optimal security design under asymmetric information, which was first asked by Myers and Majluf (1984). Over the last decades, a large literature analyzed this classic question and its variations.[1]

With few exceptions, existing literature assumes that buyers of the security are confident about the nature of asymmetric information in the sense of holding a given prior about possible distributions of cash flows (typically, ranked by some stochastic order) that the project may have. An example is that it is common knowledge that the project's cash flows follow a log-normal distribution, whose mean is privately known by the issuer. While plausible in some settings, such as when a mature firm undertakes a project similar to a project this or other firms took in the past, this assumption can be unrealistic in other settings, such as when a young firm raises financing for a project that has few close comparables. A better description of these settings can be that the investor has some (possibly, vague) idea about possible cash flow distributions that the project may take, but lacks confidence to assign specific priors to them. Instead, she takes a robust approach to evaluating securities in the sense of having a strong preference for securities that are "robust" to the investor's misspecification of the project's cash flows.[2] The goal of this paper is to develop a theory of security design under asymmetric information in this setting.

Formally, we study the following model. The issuer has some resources and raises extra to finance his project. The issuer has private information about the distribution of cash flows. In contrast, the investor only knows that a certain set of distributions of the project's cash flows is possible, called the *uncertainty set*. It captures all distributions within the neighborhood (in the sense of total variation distance) of some base distribution. Importantly, we do not impose any further restrictions on the uncertainty set, and in particular, distributions in this

[1] An incomplete list of papers includes Brennan and Kraus (1987), Nachman and Noe (1994), DeMarzo and Duffie (1999), Fulghieri and Lukin (2001), DeMarzo (2005), Fulghieri et al. (2015), Yang (2015), Bond and Zhong (2016), Szydlowski (2017), Yang and Zeng (2017).

[2] Courtney et al. (1997) give the following recommendation to practitioners about the decision making under great uncertainty: In uncertain environments when "it is impossible ... to define a complete list of scenarios and related probabilities, it is impossible to calculate the expected value of different strategies. However, establishing the range of scenarios should allow managers to determine how robust their strategy is, identify likely winners and losers, and determine roughly the risk of following status quo strategies."

set are not ordered by some stochastic ordering. The uncertainty set can come from certain distributions being discarded by the investor (e.g., based on his analysis of the project) as not possible or sufficiently unlikely to neglect them in the decision. The size of the uncertainty set reflects the degree of the investor's uncertainty: When the uncertainty set is larger, the investor entertains more possible cash flow distributions and in this sense is more uncertain about the project.

While the investor believes that the distribution of the project's cash flows is some point in this uncertainty set, she lacks confidence about which one. Formally, she has infinitely many priors ("models of the world") where each prior puts probability one on a specific point in the uncertainty set. After the investor observes the security offered by the issuer, she re-evaluates ("tests") whether the security offer can be justified in each model and keeps only the models that can justify it, denoted the *set of justifiable models*. Given it, the investor demands robustness: she evaluates the security according to the *worst-case justifiable model*, i.e., the justifiable model that yields the lowest expected value of the security. Investor's preference for robustness can be viewed as the ambiguity aversion, as in Gilboa and Schmeidler (1989). Among other things, this preference for robustness reflects the Ellsberg paradox (Ellsberg (1961)), a finding that people prefer to take risk in situations when the odds are known than when the odds are unknown.

The "test" that the investor conducts to determine if the model can justify the security offer is similar in spirit to the Intuitive Criterion in Bayesian signaling games. Specifically, the investor evaluates whether the issuer is weakly better off, if the investor accepted the observed security offer, than in equilibrium. The set of justifiable models consists of all distributions of cash flows in the uncertainty set for which the answer is a "yes".

We characterize the equilibria of this non-Bayesian signaling game. Our first result is that the equilibrium is generically unique despite both types (distributions of cash flows) and signals (security offers) being multi-dimensional and weak restrictions on the structure of the uncertainty set. This contrasts sharply with Bayesian multi-dimensional signaling models, where severe multiplicity of equilibria is common place and general characterization is elusive. The key to our generic uniqueness is the robust approach of the investor to valuation of securities. Intuitively, because securities are evaluated according to the worst-case scenario, they are priced by the investor similarly on and out of the equilibrium path, which prevents the punishment of deviations by adverse beliefs that normally sustains a variety of equilibria in the Bayesian model.

The implications of our non-Bayesian model differ significantly from the standard Bayesian models of security design. In an important paper, Nachman and Noe (1994) deliver a stark result: signaling with securities is generally rather limited, and under certain conditions, risky debt is optimal and minimizes losses from mispricing. When investors demand robustness, signaling is richer, and generally, there is partial pooling and several securities are offered in equilibrium. The type of financing depends crucially on the degree of the investor's uncertainty, or specifically, on whether the investor entertains the possibility of negative net-present value (NPV) projects or not.

Our main results can be summarized as follows. If uncertainty is small in the sense that the project has a positive NPV for any distribution in the uncertainty set, then the investor evaluates any security according to the same model, which is the distribution of cash flows skewed maximally towards low realizations. In this case, risky debt arises as an equilibrium security for any issuer's type that dominates this "worst" point of the uncertainty set in the sense of monotone likelihood ratio property (MLRP). More interestingly, if uncertainty is large, i.e., there are points in the uncertainty set for which the project has a negative NPV, then standard equity arises as an equilibrium security for some issuer's types. Furthermore, when either the uncertainty set becomes sufficiently large (the investor considers more models possible) or the larger fraction of project types in the uncertainty set has negative NPV, the standard equity becomes a dominant source of financing: All types in the uncertainty set that choose to finance the project end up raising capital with the standard equity.

The key to the distinction between the high and low uncertainty environments is the interaction between investors' demand for robustness and learning from the security offer. When uncertainty is low, the signaling role of the security offer is limited, since the worst-case justifiable model is the same for any security offer. Moreover, when the investor's worst-case justifiable model and the actual distribution of cash flows are MLRP-ordered, it is cheaper for the issuer to pay in the low states, which he considers relatively less likely compared to investors, and debt emerges as the optimal security.

When the uncertainty is large, the signaling aspect of the security offer becomes important. When negative NPV types of projects are possible, the issuer chooses a security offer to signal that the project has a non-negative NPV. For example, under equity financing, the interests of the issuer and investors are partially aligned, and in particular, equity can be a credible way for the issuer to signal that his project has non-negative NPV and is worth financing in the first place. Similarly, a sufficiently high level of debt can be a credible signal of the non-negative NPV.

If the issuer's offer credibly signals that the project has non-negative NPV, then the worst-case scenario for investors is no longer the distribution of cash flows that is maximally skewed to the lowest realizations of cash flows, but rather it is one of many distributions in the uncertainty set with zero NPV. Since there are many such distributions, now the worst-case justifiable model varies with the type of security offered. For concave securities, such as debt, which are more valuable when cash flows are more concentrated around mean, the worst case scenario is the most dispersed distribution of cash flows among distributions with zero NPV. On the contrary, for convex securities, such as call option, the worst case scenario is the most concentrated distribution among distributions with zero NPV.

This causes a discontinuous shift in the investor's worst-case justifiable model from most concentrated zero-NPV model to most dispersed when the issuer switches from convex to concave securities. This shift causes the drop in the valuation of concave securities, but does not affect equity. As a result, the equity is now optimal for a range of types. When the uncertainty increases or more projects become negative NPV, the change in the investor's worst-case justifiable model as one moves from convex to concave securities is larger, and hence, more types of issuer prefer equity, making it eventually the dominant source of financing.

Interestingly, the nature of the private information is important for the optimal security design. We consider the version of the model with the uncertainty about assets in place rather than cash flows from the new project. In this variation, when the worst-case justifiable model and the issuer's type are strictly ordered by the MLRP, the risky debt is optimal *irrespective* of the level of uncertainty, while equity is never an optimal security. When private information is about assets in place, there is an adverse selection problem: the issuer with low quality assets has a stronger preference to pledge them rather than the issuer with high quality assets. Thus, the worst case scenario for investors is always the distribution that is maximally skewed towards low realizations of cash flows, and by the same logic as in the baseline model with small uncertainty, risky debt emerges in equilibrium.

The paper has three implications. First, it shows that both standard risky debt and standard equity, i.e., two extremely popular financial contracts, arise as equilibrium outcomes from the same model of financing with only one friction – asymmetric information. Bayesian models of security design under asymmetric information often generate risky debt as the equilibrium security under some conditions, but when these conditions are violated, the equilibrium security is usually not standard equity — for example, it is common to have the opposite of risky debt, a call option, as the optimal security. For this reason, many papers that "operationalize" models of financing under asymmetric information restrict attention to debt and equity (e.g., Hennessy et al. (2010), Fulghieri et al. (2015)). In our model, the advantage of standard equity over any other security is that it, roughly speaking, minimizes the informational advantage of the issuer coming from knowing the exact form of distribution that achieves each level of the NPV.

Second, while this goes beyond the scope of the paper, our results suggest the following information-based theory of dynamic capital structure. Young firms have little assets in place, and investors face a lot of uncertainty about cash flows from their projects. As a consequence, young firms use outside equity as the source of external finance. As time goes by, they accumulate assets in place and the uncertainty about cash flows from their projects declines, as investors get enough data observations to discard some models as not plausible. For both reasons, risky debt becomes a better security to address information asymmetry problems, implying that the standard pecking order theory should be more applicable for mature firms, where there is little uncertainty and information asymmetry is primarily about assets in place.

This leads to a possible interpretation of some contradictory evidence on the validity of the pecking-order theory of financing. While pecking order works best for large mature firms (Shyam-Sunder and Myers (1999)), it does a poor job at describing financing decisions of small high-growth firms (Frank and Goyal (2003), Leary and Roberts (2010)), even though there is plausibly more information asymmetry about the latter. However, these findings can be consistent with security design implications in our paper.

Related Literature The paper is related to several strands of literature. First, as we already mentioned, we contribute to the literature on optimal security design under asymmetric information, started by Myers and Majluf (1984) (see footnote 1 for an incomplete list of papers). The formulation of our basic model is closest to Nachman and Noe (1994): like they, we consider the problem where all private information is about the investment project, and we impose the same restrictions on admissible securities. Our assets in place model is closer to DeMarzo and Duffie (1999), except that the issuer chooses the security *after* observing private information. The novelty of our setup is two-fold. First, instead of cash flow distributions all belonging to a certain class where issuer's "type" captures ranking in this class, the investor believes that "anything can happen" within some neighborhood of the base distribution. Thus, we put minimal structure on the set of distributions of cash flows that the investor considers possible.[3] The second novelty is the robust approach to

[3] At the same time, we want to stress that none of our results are driven by some exotic distributions that we allow for by relaxing the assumption about the structure of the uncertainty set. In fact, our primary

security pricing: investors evaluate securities by the worst-case justifiable model.[4] These novelties lead to results that are different from existing literature. Boot and Thakor (1993) show how riskless debt and equity can be rationalized, where the advantage of equity over risky debt is that it provides higher incentives for traders to get informed. In our model, we rule out safe debt by assuming that the lowest cash flow realization is zero.[5]

Several papers study security design problems with heterogeneous beliefs. Garmaise (2001) studies the problem when investors have diverse beliefs. Boot and Thakor (2011) studies how disagreement between the firm's initial owners and managers over project choice interacts with the firm's seurity issuance and allocation of control rights. Ortner and Schmalz (2016) study a problem of asset-backed security design when the issuer and investors disagree about probability distributions of different cash flow realizations. Kondor and Koszegi (2017) study a model in which competitive issuers design securities to sell to naive investors. In equilibrium of our model, the issuer and the investor also end up having heterogeneous beliefs, as they use different models to evaluate securities. The novelty of our model is that it results in the *endogenous* heterogeneity of beliefs: security design has a signaling role, like it does in standard Bayesian signaling models.[6]

Second, the paper is related to the growing literature on robust contracting. In this literature, the most related papers are models that study contracting in the presence of moral hazard under risk neutrality and limited liability. In Carroll (2015) and Antic (2015), the principal does not know what actions are available to the agent and demands robustness. Carroll (2015) shows that the optimal contract is linear in the setting when Knightian uncertainty of the principal is extreme. In Antic (2015), Knightian uncertainty is not extreme, so our setting is closer to his model. The conceptual difference is that these are moral hazard problems, while we study the adverse selection (signaling) problem. In other words, Carroll (2015) and Antic (2015) can be viewed as robust versions of Innes (1990), while ours can be viewed a robust version of Myers and Majluf (1984) and Nachman and Noe (1994). There are major differences in implications, but we postpone a detailed discussion until Section 7.

interest is in how the optimal financing changes with changes in the uncertainty set, i.e., as it becomes larger/smaller or includes more/less negative NPV projects.

[4]In Section 7, we discuss in more details the relationship between our approach and the classic Bayesian approach.

[5]If instead we assumed that the lowest realization is positive, then the issuer would issue as much safe debt as possible and then will issue a security prescribed by the equilibrium of our current model. However, because we have only one investor rather than many, our paper would not provide any insights about whether these claims should be combined into one or separate, which is one of the interesting insights of Boot and Thakor (1993).

[6]Less related, several papers study security design with investor's private information (Axelson (2007), DeMarzo et al. (2005), Gorbenko and Malenko (2011)).

Several other papers on robust contracting are also related. Lee and Rajan (2017) consider a robust version of Innes (1990) modeling ambiguity aversion of the principal by multiplier preferences towards model uncertainty, as in Hansen and Sargent (2001).[7] Lee (2017) incorporates manager's ambiguity aversion into a "trade-off" theory of capital structure. Zhu (2015) uses an investor with a preference for robustness to provide a micro-foundation for refinanceable contracts, including refinanceable debt. Chassang (2013) and Miao and Rivera (2016) study dynamic agency problems when the principal demands robustness. Bergemann and Schlag (2011) study monopoly pricing with the monopolist uncertain about the demand.

Third, we contribute to the literature on signaling with mulidimensional types and signals. In the Bayesian model, the characterization of the equilibrium set remains a hard, open question. Quinzii and Rochet (1985), Engers (1987) provide sufficient conditions for separating equilibria. We propose an alternative robust approach to the classic signaling model, which proves extremely tractable and allows for the complete characterization of the generically unique equilibrium. In this respect, we are close to Carroll (2016) who applies the robust approach to the multidimensional screening model, another open problem in the Bayesian formulation.

Finally, Dicks and Fulghieri (2015, forthcoming), Garlappi et al. (2016) study the role of ambiguity in other corporate finance decisions: allocation of control rights, bank runs, and a group decision about investment, respectively. An innovation of our approach is the introduction of a natural updating of models by a party with a preference for robustness. In this respect, our work is related to the literature on belief updating by ambiguity-averse agents (most closely to Epstein and Schneider (2007)).

The structure of the paper is as follows. Section 2 describes the signalling game. Section 3 shows that the equilibrium is generically unique. Section 4 analyzes the model with three realizations of future cash flows. Section 5 studies the version of the model with assets in place. Section 6 provides general results. Section 7 discusses the existing empirical evidence, the relation to Bayesian signaling models. Section (8) shows robustness of our results. Section 9 concludes and gives directions for future research. Key proofs are presented in the text, the rest are relegated to Appendix and Online Appendix.

[7] Hébert (forthcoming) considers a version of Innes (1990) in which the agent has a flexible moral hazard technology. However, there is no preference for robustness.

2 The Model

This section describes the model. In Section 2.1, we introduce the capital raising game between the informed issuer and the investor facing Knightian uncertainty about the distribution of the project's cash flows. We proceed by defining the equilibrium. In Section 2.2, we introduce the notion of a set of justifiable models, which specifies how the investor learns from observing the issuer's offer.

2.1 Model Setup

The issuer (male) has a project that requires investment K. The issuer has $W < K$ of his own resource and needs to complement it by raising financial capital $I \equiv K - W$ from the outside investor (female). Several interpretations of W are possible. If the firm already operates, then it is natural to interpret W as the firm's assets in place and the issuer as the firm's management operating the firm in the interest of current shareholders (Myers and Majluf (1984)). In the case of a newly created firm, it is natural to think about the issuer as the entrepreneur and about W as the entrepreneur's outside option, such as the value from an alternative employment that he foregoes by undertaking the project.

Information Structure The issuer knows the distribution f of the future cash flow from the project. We call f the *type of the issuer's project*, or simply, *the issuer's type*. The future cash flow z can take one of $N+1$ values in $Z = \{z_0, z_1, \ldots, z_N\}$, where $z_0 = 0 < z_1 < \cdots < z_N$. For any f, we denote probabilities of $z_0, z_1, \ldots, z_N$ by $f_0, f_1, \ldots, f_N$, respectively, and associate type f with point $(f_1, \ldots, f_N)$ in the probability simplex $\Delta(Z) \equiv \{f \in \mathbb{R}_+^N : \sum_{n=1}^N f_n \leq 1\}$. Denote by F the c.d.f. of distribution f.

The investor faces Knightian uncertainty about distribution f. We capture it by a set of distributions $B \subset \Delta(Z)$, referred to as the *uncertainty set*. It includes all distributions in the neighborhood of some base distribution $g = (g_0, ..., g_N)$, satisfying $g \in \mathbb{R}_+^N$ and $\sum_{n=1}^N g_n \leq 1$. To define set B, we need to choose a specific metric to quantify the distance between two probability distributions. We focus on neighborhoods induced by the total variation distance, because it is one of the most widely used probability metrics and because it has a natural interpretation. Specifically, set B is the set of all distributions whose total variation distance from base distribution g does not exceed ν:

$$B = \left\{ f \in \Delta(Z) : \sup_{A \subset Z} |\mathbb{P}_f(A) - \mathbb{P}_g(A)| \leq \nu \right\}, \tag{1}$$

where A is any measurable event (any subset of Z) and $\mathbb{P}_f(A)$ and $\mathbb{P}_g(A)$ are the probabilities of it occuring under distribution f and base distribution g, respectively. Total variation distance is a natural notion of "closeness" of distributions capturing that two distributions are close if they assign sufficiently similar probabilities (different at most by ν) to any event. Because the state space is countable, definition (1) is equivalent to (e.g., Huber (2011)):

$$B = \left\{ f \in \Delta(Z) : \sum_{n=0}^{N} |f_n - g_n| \leq 2\nu \right\}. \tag{2}$$

We will refer to ν as the *degree of (investor's) uncertainty.* The larger ν, the more uncertain the investor in the sense that she considers more distributions of cash flows as possible.[8]

To have a non-trivial problem, we assume that the project has positive NPV for at least one point in set B: $\{f \in B : \mathbb{E}_f[z] \geq K\} \neq \emptyset$. If this condition is violated, then the investor believes that the project has a negative NPV, so, as it will be clear from what follows, the project does not get financed for any $f \in B$.

Figure 1 illustrates set B in the case of $N = 2$. In this case, B can be parametrized by f_1 and f_2 both in $[0, 1]$ that satisfy the following constraints $f_0 \in [g_0 - \nu, g_0 + \nu], f_1 \in [g_1 - \nu, g_1 + \nu], f_2 \in [g_2 - \nu, g_2 + \nu]$. Set B has a natural interpretation: the investor has a reference distribution of cash flows, g, but allows for a possibility that she knows the probability of each realization $z \in \{0, z_1, z_2\}$ not exactly, but with some error ν.[9]

Timing and Actions The timing of the game is as follows:

1. The issuer makes an offer to the investor of security $s = (s_0, s_1, ..., s_N)$ that pays s_n if the cash flow realization is z_n. The issuer can also choose not to pursue the investment, which we refer to as offering security $s = \mathbf{0}$, which pays zero for any cash flow realization. In this case, the issuer's payoff is W, and the investor's payoff is zero.

2. Having observed the security offer s, the investor chooses $\sigma \in \{0, 1\}$ whether to invest I in exchange for security s ($\sigma = 1$) or not ($\sigma = 0$).

3. If the investor accepts the offer ($\sigma = 1$), the investment is made, and the cash flow

[8]While our model assumes that set B is induced by the total variation distance, our main results do not rely on the specific form of B. In Section 6, we show that many of our results are not sensitive to the specific form of B, as long as B is convex with a non-empty interior. Furthermore, when $N = 2$, many probability metrics result in the same set B: in particular, Kolmogorov, Levy, and Prokhorov metrics result in exactly the same set B as the total variation distance, provided that possible realizations of z are sufficiently far apart.

[9]Note that $f_0 \in [g_0 - \nu, g_0 + \nu]$ is equivalent to $f_1 + f_2 \in [g_1 + g_2 - \nu, g_1 + g_2 + \nu]$.

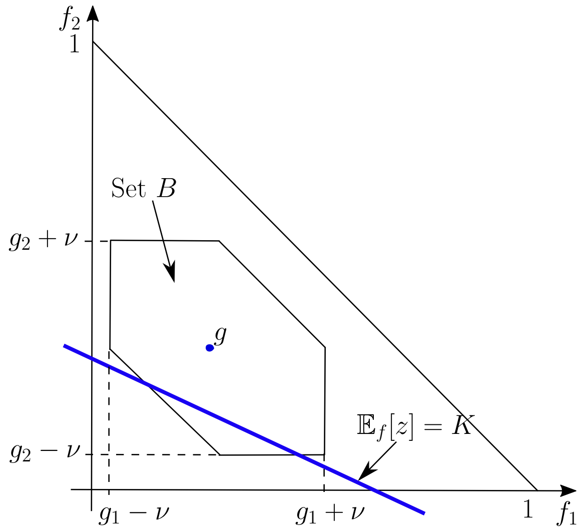


Figure 1: ***Illustration of the uncertainty set*** B
The uncertainty set B *is the neighborhood of radius* ν *around the base distribution* $\mathbf{f}$*. The blue, bold line represents types* f *satisfying* $\mathbb{E}_f[z] = K$*.*

from the project $z \in \{z_0, ..., z_N\}$ is realized. The issuer gets $z - s$ and the investor gets s (i.e., the net payoff of $s - I$). If the investor rejects the offer ($\sigma = 0$), then the investor gets zero and the issuer keeps W.

We make the standard assumption in the security design literature (e.g., Nachman and Noe (1994), DeMarzo and Duffie (1999)) that the set of feasible securities, denoted $\mathcal{S}$, consists of all monotone securities that satisfy the limited liability condition:

Definition 1. *Security* $s = (s_0, ..., s_N)$ *is* feasible *if it satisfies: (1)* $0 \leq s_n \leq z_n$ *for all* $n = 0, 1, \ldots, N$ *(limited liability); and (2)* s_n *and* $z_n - s_n$ *are weakly increasing in* n *(monotonicity). Set* $\mathcal{S}$ *is defined as the set of all feasible securities.*

The first condition states that repayment to either party must be non-negative. In particular, since $z_0 = 0$, $s_0 = 0$ for any $s \in \mathcal{S}$. The second condition states that the payoff of each party must be weakly increasing in the realized cash flow z.[10]

Equilibrium The *issuer's pure strategy* $s^*(\cdot)$ maps any $f \in B$ into a feasible security $s^*(f) \in \mathcal{S}$ that the issuer's type f offers to the investor. The *investor's pure strategy* $\sigma^*(s)$ is a mapping from any security $s \in \mathcal{S}$, $s \neq \mathbf{0}$ into the decision whether to accept ($\sigma^*(s) = 1$) or reject ($\sigma^*(s) = 0$) it.

[10]Monotonicity can be justified by a "sabotage" argument: if a security were non-monotone, one of the parties would be better off destroying some output for some realizations of z. See, e.g., Hart and Moore (1995).

The key feature of our model is that the investor faces Knightian uncertainty over the distribution f. In the case of Bayesian uncertainty, the investor has a single "model of the world," represented by a prior belief $\mu \in \Delta(B)$. In contrast, we assume that the investor lacks confidence in assigning prior beliefs to distributions in B. Formally, she has infinitely many "models of the world". Each model is a degenerate distribution that puts probability one on the particular distribution $f \in B$.[11] Thus, we can identify the set of models (i.e., the set of degenerate distributions with support in B) with the uncertainty set B, and use terms "model f" and "distribution f" interchangeably. After receiving a security offer s, the investor reevaluates the set of models B into a subset $B(s) \subseteq B$ of *justifiable models.* Specifically, for each model $f \in B$, the investor runs a test, which we specify in the next subsection, determining whether offer s can be justified in model $f \in B$, and keeps only models that pass the test. We refer to $B(\cdot) : \mathcal{S} \to 2^B$ as the *model updating mapping* that maps securities in $\mathcal{S}$ into subsets of B.

Having reevaluated the set of models, the investor demands robustness in the sense that given security offer s and the set of justifiable models $B(s)$, the investor evaluates the security by the justifiable model that yields the lowest value of the security. Formally, the investor values security $s \in \mathcal{S}$ at[12]

$$P(s) \equiv \min_{f \in B(s)} \mathbb{E}_f[s]. \tag{3}$$

The investor's preference for robustness can be interpreted via a game played between the investor and adversarial nature: The investor believes that after she accepts security offer s, the adversarial nature will pick $f \in B(s)$ with the objective of minimizing the investor's payoff. This game has two conceptual differences from the standard moral hazard problem: first, the adversarial nature has no incentive compatibility constraint to be satisfied; second, the set of actions that the nature can take is affected by the issuer's security choice. In what follows, we call the minimization problem in (3) the nature's problem.

We refer to the distribution that solves program (3) as the *worst-case justifiable model* and denote it by $f^*(s)$.[13] We call $P(s)$ the *valuation* of security s by the investor. Then,

[11] One can define the model more generally as any distribution over the issuer's types in B. Denote the set of such models by $\hat{B}$, and by $\hat{B}(s)$ the set of all distributions over $B(s)$. This more general definition would not affect our results, as it can be easily verified that the minimum in (4) when $f \in \hat{B}(s)$ is always attained by some distribution that puts probability one on some distribution in B (i.e., $f(s) \in B(s)$).

[12] Given the definition of $B(s)$ in the next subsection, $B(s)$ is a compact subset of B, so the minimum in (3) is attained.

[13] If there are multiple f that solve (3), all of them yield the same investor's expected utility, and we specify $f^*(s)$ to be any arbitrary selection from the solution to (3).

the investor's utility from investing I in exchange for s equals

$$V(s) \equiv P(s) - I = \min_{f \in B(s)} \mathbb{E}_f [s - I]. \tag{4}$$

Utility in (4) coincides with Gilboa and Schmeidler's (1989) maxmin expected utility representation of ambiguity aversion. However and importantly, the issuer's security choice puts a restriction on the set of distributions over which the expected value of the security is minimized.

We next define the equilibrium:

Definition 2. *A pair of strategies* $(s^*(\cdot), \sigma^*(\cdot))$ *and a model updating mapping* $B(\cdot)$ *constitute* an equilibrium *if*

1. *For any* $f \in B$,

$$s^*(f) \in \arg\max_{s \in \mathcal{S}} \{\sigma^*(s)(\mathbb{E}_f[z - s] - W)\},$$

   with $s^*(f) = \mathbf{0}$ *for any* f *such that* $\max_{s \in \mathcal{S}: \sigma^*(s)=1} \{\mathbb{E}_f[z - s] - W\} < 0$.

2. *For any* $s \in \mathcal{S}$, $\sigma^*(s) = 1$ *if and only if* $P(s) \geq I$, *where* $P(\cdot)$ *is given by* (3).

3. *For any* $s \in \mathcal{S}$, $B(s)$ *is a set of justifiable models, defined in subsection* 2.2 *below.*

The first condition is rationality of the issuer: Given his known distribution of cash flows f and the equilibrium action of the investor $\sigma^*(s)$ for each security $s \in \mathcal{S}$, the issuer chooses the security to maximize his expected payoff.[14] Note that by definition of the equilibrium, any type of the issuer that in equilibrium does not get financing offers zero security $s = \mathbf{0}$, and thus any non-zero security offered by some type in equilibrium gets accepted. We denote the set of these securities by $\mathcal{S}^* \equiv \bigcup_{f \in B: \sigma^*(s^*(f))=1} \{s^*(f)\}$. The second condition states that the investor accepts any security that she values weakly above her investment I and rejects any security she values below that. Note that the security valuation $P(\cdot)$ in the expression (3) reflects the investor's preference for robustness. Finally, the last condition of Definition 2 requires the investor's learning from observing the issuer's offer s to be "reasonable." Next, we define what we mean by "reasonable."

[14] As we will show below, in equilibrium, only a Lebesgue measure zero of types is indifferent between issuing some security and not making an offer ($s = \mathbf{0}$), and so, it is essentially without loss of generality to suppose that the issuer prefers to issue some security to not issuing when indifferent. We break ties in the investor's decision in favor of acceptance of the offer to guarantee that the maximum in the issuer's problem is indeed attained.

2.2 Learning under Knightian Uncertainty

The offer of the issuer potentially conveys information to the investor about the distribution of the project's cash flows. When the investor observes security s, she rules out some models $f \in B$ as implausible and keeps only subset $B(s)$ of models she could justify. We define $B(s)$ as:

Definition 3. *Fix $s^*(f)$ and let $U^*(f)$ be the issuer's expected utility (net of W) if it offers $s^*(f)$:*

$$U^*(f) = \begin{cases} \mathbb{E}_f [z - s^*(f)] - W, & \text{if } s^*(f) \in \mathcal{S}^*, \\ 0, & \text{otherwise.} \end{cases}$$

The model updating mapping $B(\cdot)$ is justifiable *if*

$$B(s) = \{f \in B : \mathbb{E}_f [z - s] - W \geq U^*(f)\}, \tag{5}$$

whenever this set is non-empty. If it is empty, then $B(s) = B$.

This definition is critical for the results of the paper, so it is worth describing it in detail. For each model $f \in B$, the investor runs a test whether she can justify the issuer offering security s given $f \in B$. In this test, the investor asks the following question for any $f \in B$:

> "If I accepted offer s, would the issuer be weakly better off than if he instead issued an equilibrium security $s^*(f)$ or chose not to invest in the project entirely?"

If $f \in B$ satisfies $\mathbb{E}_f [z - s] \geq \max \{\mathbb{E}_f [z - s^*(f)], W\}$, i.e., if condition ((5)) holds, then the answer to this question is a "yes", meaning that the investor can justify the observation of s when $f \in B$. Otherwise, the issuer cannot justify it. The set of justifiable models, $B(s)$, consists of all models $f \in B$ that pass this test. If security s is such that no model $f \in B$ passes this test, then the investor does not learn anything and believes that all models from set B are plausible.

The idea behind our specification of the set of justifiable models is that the investor tries to learn about the distribution of cash flows from the fact that the issuer wants to undertake the project keeping the residual security. One can see the parallel between (5) and the Intuitive Criterion of Cho and Kreps (1987) for Bayesian signaling games. According to the Intuitive Criterion, the receiver cannot rationalize a sender type to send a certain signal if this type can only do worse with this signal than in equilibrium. If some types can be rationalized while others cannot, the receiver's belief must place positive probability only on the former set of types. Condition (5) is essentially the same test as the Intuitive Criterion: Similar to how the Intuitive Criterion would require the investor to put zero belief on types that violate (5), our definition of the set of justifiable models requires the investor to "discard" models that violate (5). Therefore, our updating rule can be viewed as the Intuitive Criterion applied to the game where the investor faces multiple degenerate priors instead of holding one non-degenerate prior.

As we shall see after we solve the model, for any security $s \in \mathcal{S}^*$, i.e., for any security that some issuer's type $f \in B$ offers in equilibrium, our model updating rule becomes very similar to the model of learning under ambiguity introduced by Epstein and Schneider (2007). In Section 7, we discuss alternative models of learning from security offers.

3 Equilibrium Valuation of Securities

The game we formulated looks potentially intractable because the valuation of each security $P(s)$ depends on the as-yet-unknown equilibrium via set $B(s)$. Furthermore, because there are generally multiple equilibria in Bayesian signaling models, it can be natural to expect our game to have multiple equilibria too. In this section, we give two technical results that simplify the valuation problem considerably. Then, we use these results to show that the equilibrium is generically unique.

The first result shows that it is without loss of generality for the equilibrium analysis to restrict attention to securities that make the investor indifferent, i.e., securities that satisfy $P(s) = I$. Intuitively, there is no value for the issuer to offer the investor more than she requires. If the investor more than breaks even at security s, the issuer can also get the investor's acceptance by offering a security that is uniformly marginally worse than s. Formally:

Lemma 1. *For any $s \in \mathcal{S}$ such that $P(s) > I$, there exists $\gamma \in (0,1)$ such that $P(\gamma s) = I$. In particular, $P(s) = I$ for any $s \in \mathcal{S}^*$.*

Lemma 1 follows from the maximum theorem. The key observation in the proof is that although the worst-case justifiable model $f^*(s)$ may vary discontinuously with s, the resulting investor's valuation is a continuous function of s, except, possibly, when the set of justifiable models changes discontinuously from ((5) to B, which occurs when no model in B passes the test.

Lemma 1 has two implications. First, to verify that a certain strategy profile $\{s^*(f), f \in B\}$ is an equilibrium issuer's strategy, it is sufficient to show that no type of the issuer benefits from a deviation to securities satisfying $P(s) = I$. Intuitively, if some type of the issuer benefited from a deviation to security $s : P(s) > I$, then he could lower the security uniformly by factor γ, so that the investor is promised just enough to accept it, and benefit even more. Second, we can re-state the issuer's problem of maximizing expected payoff in terms of the minimization of mispricing. To see this, for any s such that $P(s) = I$ (which can potentially be a tempting deviation for the issuer), the issuer's payoff can be written as:

$$\mathbb{E}_f[z - s] + I - K = \underbrace{\mathbb{E}_f[z] - K}_{\text{NPV}} - \underbrace{(\mathbb{E}_f[s] - P(s))}_{\text{Mispricing}}.$$

The issuer's expected payoff consists of the project's NPV and the mispricing term arising because the investor prices the security according to the worst-case model in $B(s)$ rather than the true distribution f. Putting together these two implications, we can find any equilibrium of our game by solving the following program. For any $f \in B$, we first solve

$$\min_{s \in \mathcal{S}} \{\mathbb{E}_f[s] - P(s) \text{ s.t. } P(s) = I\} \tag{6}$$

That is, each issuer type f determines the set of securities, call it $S^*(f)$, that minimize the mispricing. If the value of the program (the minimal mispricing for type f) is greater than the NPV of the project $\mathbb{E}_f[z] - K$, then in equilibrium, the issuer type f does not issue any security. Otherwise, the equilibrium security $s^*(f)$ is contained in $S^*(f)$.

However, we still face the difficulty that the value of each security depends on the as-yet-uknown set $B(s)$. The next lemma is the central result of the section. It shows that the value of any security, which can be potentially relevant for the analysis (i.e., satisfying $P(s) = I$), can be calculated without the knowledge of $B(s)$. Let us introduce the following subsets of B:

$$\begin{aligned} B_+ &\equiv \{f \in B : \mathbb{E}_f[z] \geq K\}, \\ B_0 &\equiv \{f \in B : \mathbb{E}_f[z] = K\}, \\ B_- &\equiv \{f \in B : \mathbb{E}_f[z] < K\}. \end{aligned}$$

Thus, B_+, B_0, and B_- are the sets of, respectively, non-negative, zero, and negative NPV projects in B. When $B_+ = B$, the investor believes that the project is definitely positive-NPV, while when $B_+ \subset B$, the investor entertains the possibility that the project is negative-NPV. Equipped with these definitions, we state our main technical result:

Lemma 2. *For any security $s \in \mathcal{S}$ such that $P(s) = I$, it holds $f^*(s) \in \arg\min_{f \in B_+} \mathbb{E}_f[s]$ and*

$$P(s) = \min_{f \in B_+} \mathbb{E}_f[s]. \tag{7}$$

Proof. We first show that $f^*(s) \in B_+$. By the definition of $B(s)$, $f^*(s) \in B(s)$ implies $\mathbb{E}_{f^*(s)}[z - s] \geq W$. If $\mathbb{E}_{f^*(s)}[z] < K$, then $P(s) = \mathbb{E}_{f^*(s)}[s] \leq \mathbb{E}_{f^*(s)}[z] - W = \mathbb{E}_{f^*(s)}[z] - K + I < I$, which contradicts to $P(s) = I$. Thus, $f^*(s) \in B_+$.

Now, suppose that for some $s \in \mathcal{S}$, it holds that $P(s) = I$, but $\mathbb{E}_{f^*(s)}[s] > \mathbb{E}_{\tilde{f}}[s]$ for some $\tilde{f} \in \arg\min_{f \in B_+} \mathbb{E}_f[s]$. Observe that $\tilde{f} \in \arg\min_{f \in B_+} \mathbb{E}_f[s]$ implies that type $\tilde{f}$ issues some security $\tilde{s} = s^*(\tilde{f})$ in equilibrium.[15] We will show that type $\tilde{f}$ prefers to deviate to s in this case. Since $P(s) = I$ and $P(\tilde{s}) = I$ (by Lemma 1), it is sufficient to show that the mispricing of s ($\mathbb{E}_{\tilde{f}}[s] - P(s)$) is smaller than the mispricing of $\tilde{s}$ ($\mathbb{E}_{\tilde{f}}[\tilde{s}] - P(\tilde{s})$). The mispricing from issuing security $\tilde{s}$ for type $\tilde{f}$ equals

$$\mathbb{E}_{\tilde{f}}[\tilde{s}] - P(\tilde{s}) \geq \min_{f \in B(\tilde{s})} \mathbb{E}_f[\tilde{s}] - P(\tilde{s}) = 0,$$

where the inequality holds from $\tilde{f} \in B(\tilde{s})$ and the equality holds from the pricing of $P(\tilde{s})$. At the same time, the mispricing from issuing security s for type $\tilde{f}$ equals $\mathbb{E}_{\tilde{f}}[s] - P(s) < \mathbb{E}_{f^*(s)}[s] - P(s) = 0$. Since $P(s) = P(\tilde{s}) = I$, these two inequalities imply that $\mathbb{E}_{\tilde{f}}[s] < \mathbb{E}_{\tilde{f}}[\tilde{s}]$. Thus, type $\tilde{f}$ is better off deviating to issuing s, which contradicts the premise that type $\tilde{f}$ issues $\tilde{s}$ in equilibrium. □

Lemma 2 has three implications. First, it shows that for all relevant securities, the equilibrium pricing can be determined without the knowledge of $B(s)$ and is given by (7). In other words, in solving the Nature's problem, it is sufficient to minimize $\mathbb{E}_f[s]$ over the set B_+, and one can abstract from the particular shape of $B(s)$. (However, note that this does not mean that $B(s) = B_+$.) This implies that we can rewrite the program (6) as:

$$\begin{aligned} &\min_{s \in \mathcal{S}} \{\mathbb{E}_f[s] - \min_{g \in B_+} \mathbb{E}_g[s]\} \\ &\text{s.t. } \min_{g \in B_+} \mathbb{E}_g[s] = I. \end{aligned} \tag{8}$$

Second, Lemma 2 implies that the issuer is able to signal that the project has positive

[15] Indeed, suppose it were not the case and type $\tilde{f}$ made no offer. For equity $s = \frac{I}{K} z$, $\mathbb{E}_f[z - s] \geq W$ if and only if $\mathbb{E}_f[s] \geq I$ and so such offer would be accepted by the investor. The expected utility of type $\tilde{f}$ from such security is $\mathbb{E}_f[z - s] - I + K = (1 - \frac{I}{K})(\mathbb{E}_f[z] - K) \geq 0$ and so, type $\tilde{f}$ weakly prefers to issue it, which is a contradiction.

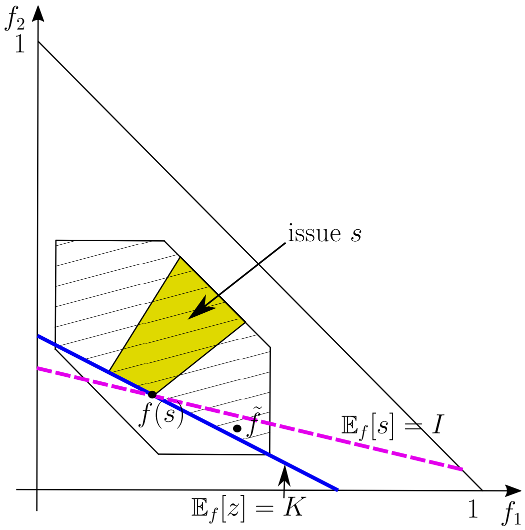


Figure 2: ***Illustration for Lemma*** 2
The hatched region is B_+, the shaded region is $B(s)$, the solid line is $\mathbb{E}_f[z] = K$, the dashed line is $\mathbb{E}_f[s] = I$. In the figure, security s issued in equilibrium by types in the shaded region is priced at $P(s) = \mathbb{E}_{f(s)}[s] = I$ above $\min_{f \in B_+} \mathbb{E}_f[s]$. (This follows from the fact that iso-line $\mathbb{E}_f[s] = I$ passing through $f(s)$ intersects the interior of set B_+). Type $\tilde{f}$ prefers to issue security s to the security $\tilde{s} = s^*(\tilde{f})$ that he issues in equilibrium, as for type $\tilde{f}$ the mispricing from security s is negative, while it is non-negative from $\tilde{s}$. Thus, this is impossible in equilibrium.

NPV and this way the equilibrium investment is always efficient. Equity financing is an example of a security that can be a credible signal that the NPV is positive. Indeed, if the issuer offers an equity stake $\frac{I}{K}$, then the interests of the issuer and investor are perfectly aligned: $\mathbb{E}_f[z - \frac{I}{K}z] + I - K \geq 0$ if and only if $\mathbb{E}_f[\frac{I}{K}z] - I \geq 0$. Another example is a sufficiently high level of debt. Indeed, consider the debt level d sufficiently high so that $\max_{f \in B_-} \mathbb{E}_f[\max\{0, z - d\}] < W$.

Third, one might expect that the issuer is able to signal some information about his type through the choice of security and this way reduce the mispricing further, i.e., $\min_{f \in B_+} \mathbb{E}_f[s] < P(s)$. Lemma 2 shows that this is not possible. To see why, consider the case of $N = 2$ depicted in Figure 2. Suppose that types in the shaded region separate in equilibrium and issue security s that is priced at $I = \mathbb{E}_{f(s)}[s] > \min_{f \in B_+} \mathbb{E}_f[s]$. Consider type $\tilde{f} \in B_+$ such that $I > \mathbb{E}_{\tilde{f}}[s]$. In equilibrium, such type issues some security $\tilde{s} = s^*(\tilde{f})$, and by Lemma 1, his mispricing is non-negative. However, if type $\tilde{f}$ issues security s, its mispricing would be negative, and thus, s would constitute a profitable deviation for type $\tilde{f}$, which is a contradiction.

There are generally multiple equilibria in the Bayesian signaling models with multidimensional signals and types, unless one puts strong restrictions on the type or signal space. In contrast, in our non-Bayesian model the equilibrium is generically unique. This is formalized in the following proposition:

Proposition 1. *The equilibrium is generically unique, i.e., the set of issuer types that offer different securities in different equilibria has Lebesgue measure zero.*

Since the proof is somewhat technical, we relegate it to the appendix. It follows from the linearity of the objective function in s and sets B_+ and $\mathcal{S}$ being convex polyhedra.

The generic uniqueness in Theorem 1 is somewhat surprising in the signaling model with multidimensional types and signals. To provide some intuition, it is useful to recall the reason for multiplicity of equilibria in Bayesian signaling games. There, multiple equilibria can be sustained by the adverse inference in the case of deviation: The receiver believes that all deviations come from the "worst" type. This makes it easy to deter deviations because on-path actions are evaluated according to the posterior believes about the sender's type and all deviations are evaluated by the "worst" type. This is not the case in our model, because all securities are evaluated by their worst-case justifiable model, and in this sense, securities on and out of the equilibrium path are evaluated similarly. Thus, we no longer can use the adverse inference by the investor to sustain multiple equilibria.

In the next section, we analyze the model with three states, where we demonstrate how one can completely characterize the equilibrium

4 Model with Three States

Consider the case of three states ($N = 2$). It will be useful to define the likelihood ratio between two positive cash flows $\zeta_f \equiv \frac{f_2}{f_1}$. As we will see, there are two distinct cases depending on whether the NPV is non-negative for all $f \in B$ (i.e., $B_+ = B$) or the NPV is negative for some $f \in B$ (i.e., $B_+ \subset B$). We start with the simpler (but less interesting) case when all projects in B have non-negative NPV and continue with the more interesting case when the investor is uncertain whether the project has non-negative NPV or not. That section presents the main results of the paper.

4.1 Case 1: Small Uncertainty

We start with the simpler (but less interesting) case when all projects in B have non-negative NPV, so $B_+ = B$. Informally, the investor is certain that the project is good but is uncertain (in the Knightian sense) over the distribution of its cash flows.

Equilibrium Valuation of Securities We first consider equilibrium pricing. Because $B_+ = B$ and any security is weakly monotone, Lemma 2 implies that $f^*(s) = \underline{f}$ for any security that can be potentially relevant for the analysis. Thus, there is a unique model $\underline{f}$, independent of the security offer, at which the seller evaluates any relevant security offer. Model $\underline{f}$ is the distribution in B that is maximally skewed towards low realizations: it puts probability $\min\{g_0 + \nu, 1\}$ on $z = 0$ and probability $\max\{g_2 - \nu, 0\}$ on $z = z_2$. Figure 3 depicts $\underline{f}$. The next lemma formalizes this statement:[16]

Lemma 3. *If $\mathbb{E}_f[z] \geq K$ $\forall f \in B$, then for any $s \in \mathcal{S}$ so that $P(s) = I$, it holds $f^*(s) = \underline{f}$ and*

$$P(s) = \mathbb{E}_{\underline{f}}[s]. \tag{9}$$

Intuitively, because the investor is certain that the project has a non-negative NPV, she knows that the entrepreneur is weakly better off investing for any distribution of the project's cash flows and any security at which the investor just breaks even. Thus, distribution $\underline{f}$ is in the set of justifiable models for any security at which the investor breaks even, and, since it is also the distribution at which the value of any security is minimized, it must be the worst-case justifiable model. We use Figure 3 to provide a simple graphical argument behind Lemma 3. By Lemma 2, to find $P(s)$ we need to minimize $f_1 s_1 + f_2 s_2$ over B. Since $\frac{s_1}{s_2} \in [0, 1]$, the slope of the iso-line $f_1 s_1 + f_2 s_2$ is in $[-1, 0]$. Hence, $f_1 s_1 + f_2 s_2$ attains minimum on the south-western boundary of B, which has slope -1, and in particular, point $\underline{f}$ attains the minimum.

The key implication of Lemma 3 is that all relevant securities are priced using the most pessimistic model in B, model $\underline{f}$. As we will see in the next subsection, this is in sharp contrast with the case when the investor entertains the possibility that the project has negative NPV, where the worst-case justifiable model $f(s)$ will depend non-trivially on the security offered.

Characterization of Equilibria Given that any relevant security is priced by the investor at (9), it is straighforward to solve the issuer's problem: the issuer of type $f \in B$ simply chooses the cheapest security (i.e., with the lowest $\mathbb{E}_f[s]$) from the set of securities that satisfy $\mathbb{E}_{\underline{f}}[s]$. The following proposition characterizes the equilibrium:

Proposition 2. *Suppose that $B_+ = B$ (i.e.,$\mathbb{E}_f[z] \geq K$ for all $f \in B$). Then, in equilibrium*

[16] The proof of Lemma 3 is covered in the proof of the general Lemma 8 in Section 6 as a special case.

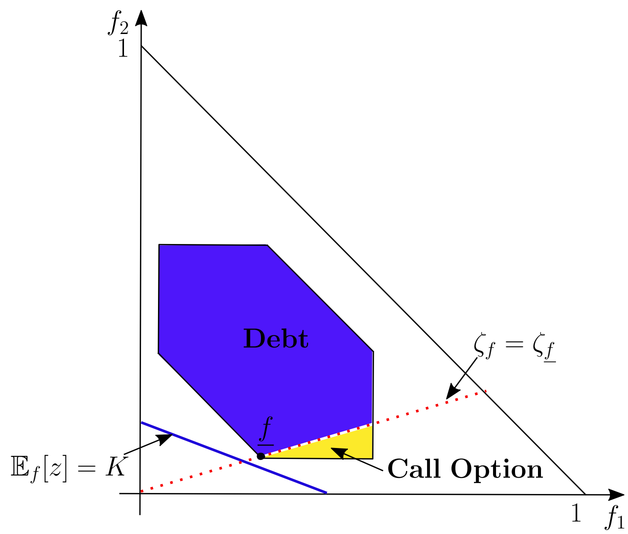


Figure 3: ***Equilibrium when uncertainty is small ($\mathbb{E}_f[z] \geq K$ for all $f \in B$)***
Distribution $\underline{f}$ is the worst-case justifiable model for any security offer with $P(s) = I$. The issuer's types in B above the dashed line assign a relatively higher probability to state z_2 relative to state z_1 and so, prefer to shift the payment to the investor maximally to state z_1. Hence, debt is optimal for them. The opposite is true for types in B below the dashed line, who issue call option in equilibrium.

1. *types f with $\zeta_f > \zeta_{\underline{f}}$ offer a standard risky debt security with face value d ($s(z) = \min\{d, z\}$) such that $\mathbb{E}_{\underline{f}}[\min\{z, d\}] = I$;*

2. *types f with $\zeta_f < \zeta_{\underline{f}}$ offer a call option with strike price k ($s(z) = \max\{z - k, 0\}$) such that $\mathbb{E}_{\underline{f}}[\max\{z - k, 0\}] = I$;*

3. *any type f with $\zeta_f = \zeta_{\underline{f}}$ offer any security s satisfying $\mathbb{E}_{\underline{f}}[s] = I$.*

Proof. By Lemma 1, we can consider only securities s such that $\underline{f}_1 s_1 + \underline{f}_2 s_2 = I$. The mispricing of type f from issuing s equals

$$\begin{aligned} \mathbb{E}_f[s] - \mathbb{E}_{\underline{f}}[s] &= (f_1 - \underline{f}_1)s_1 + (f_2 - \underline{f}_2)s_2 \\ &= \left(\frac{1}{\zeta_f} - \frac{1}{\zeta_{\underline{f}}}\right) f_2 s_1 + \left(\frac{f_2}{\underline{f}_2} - 1\right) I. \end{aligned}$$

If $\zeta_f > \zeta_{\underline{f}}$, then type f prefers to issue s that maximizes s_1 and so, issues debt in equilibrium. If $\zeta_f < \zeta_{\underline{f}}$, then type f prefers to issue s that minimizes s_1 and so, issues call option. Since $\mathbb{E}_f[z - s] \geq \mathbb{E}_{\underline{f}}[z - s]$ for all $s \in \mathcal{S}$ and $\mathbb{E}_{\underline{f}}[z - s] = W$ for $s \in S^*$, $\mathbb{E}_f[z - s^*(f)] - W \geq 0$ for all $f \in B$. □

Figure 3 illustrates the equilibrium. The dashed line depicts types that are indifferent between all securities satisfying $\mathbb{E}_{\underline{f}}[s] = I$. Types above the dashed line (with $\zeta_f > \zeta_{\underline{f}}$) issue risky debt. Intuitively, the issuer is endogenously more optimistic about the project than the investor. Furthermore, for any distribution above the dashed line, the issuer is uniformly more optimistic about higher realizations of the cash flows (in the sense of monotone likelihood ratio property (MLRP)) than the investor. Thus, the issuer is better off keeping the security that gives him the maximum upside and giving the investor the security that gives her the highest payoff in the low cash flow realizations. The security that achieves this is the standard debt security. Types below the dashed line (with $\zeta_f < \zeta_{\underline{f}}$) issue a call option. For these distributions, even though the issuer is more optimistic about the project than the investor, his optimism is not monotone in the states: he knows that the probability of the highest cash flow realization is rather low, while the probability of the medium cash flow realization is rather high. Thus, she wants to keep the security that gives her the highest payoff for medium realizations of the cash flow while selling the security that gives the investor paid in the highest cash flow realizations.

It is instructive to see how the model updating mapping $B(s)$ looks like in the (generically unique) equilibrium described in Proposition 2, and why deviations to other securities are not profitable. Consider debt securities with other face values. Clearly, the issuer of any type does not benefit from issuing debt with face value exceeding d in Proposition 2, because he can offer debt with face value d and have it accepted by the investor. If the issuer offers debt with face value $d' < d$, then the investor could justify type $\underline{f}$ to offer debt with such value. Because $\mathbb{E}_{\underline{f}}[\min\{z, d'\}] < I$, the investor will reject the offer, making it unattractive for the issuer. More generally, for any security $s \in \mathcal{S}$, either no type could be better off offering it, or if some types could be better off offering, then type $\underline{f}$ would also be better off offering it (if it gets accepted by the investor), implying that model $\underline{f}$ would be justifiable and, hence, values at less than I and thus rejected by the investor.

To sum up, when the uncertainty is small in the sense that the investor is confident that the project has a non-negative NPV, the investor evaluates all relevant securities using the same model $\underline{f}$, which is the most pessimistic model in set B. As a consequence, the issuer "typically" finances the project with standard risky debt, that is, whenever the distribution of project cash flows dominates the most pessimistic model $\underline{f}$ in the sense of MLRP. In other words, when uncertainty is small, the security design implications of the model with Knightian uncertainty are conceptually similar to the implications of the model with Bayesian uncertainty (Nachman and Noe (1994)). In both cases, risky debt arises as an equilibrium security under a stochastic dominance restriction.

4.2 Case 2: Large Uncertainty

We now turn to the more interesting case when the investor entertains the possibility that the project has a negative NPV:$\mathbb{E}_f[z] < K$ for some $f \in B$. As we will show, unlike in the small uncertainty case, now the security offer will have a non-trivial effect on the investor's worst-case justifiable model, $f^*(s)$.

Equilibrium Valuation of Securities Because some distributions in set B yield negative NPV, there exists a set B_0 of zero-NPV projects that the investor believes are possible. They are given by the intersection of the zero-NPV line $\mathbb{E}_f[z] = K$ with set B (see Figure 5 for an illustration). Let ψ and ϕ be the distributions in B_0 that assign, respectively, minimal and maximal probabilities to z_1 among all distributions in B_0:

$$\begin{aligned} \psi &= \arg\min_{f \in B} \{f_1 \text{ s.t. } f_1 z_1 + f_2 z_2 = K\}, \quad (10) \\ \phi &= \arg\max_{f \in B} \{f_1 \text{ s.t. } f_1 z_1 + f_2 z_2 = K\}. \quad (11) \end{aligned}$$

In other words, ψ is the most dispersed and ϕ is the least dispersed (in the sense of the mean-preserving spread) among possible distributions of the project's cash flows that yield zero NPV. (See Figure 5 for the illustration of ψ and ϕ). Distributions ϕ and ψ play a key role in the security pricing as the following lemma demonstrates.

Lemma 4. *For any $s \in \mathcal{S}$ such that $P(s) = I$,*

$$f^*(s) = \begin{cases} \psi & \text{if } s_1 > \frac{z_1}{z_2} s_2, \\ \phi & \text{if } s_1 < \frac{z_1}{z_2} s_2, \\ \alpha\psi + (1-\alpha)\phi, \forall \alpha \in (0,1) & \text{if } s_1 = \frac{z_1}{z_2} s_2, \end{cases} \quad (12)$$

and

$$P(s) = \begin{cases} \mathbb{E}_\psi[s], & \text{if } s_1 \geq \frac{z_1}{z_2} s_2, \\ \mathbb{E}_\phi[s], & \text{if } s_1 \leq \frac{z_1}{z_2} s_2. \end{cases} \quad (13)$$

Proof. We first show that for any $s \in \mathcal{S}$ such that $P(s) = I$, there exists $f \in B_0$ such that $P(s) = \mathbb{E}_f[s]$, which would imply that it is without loss of generality to restrict $f(s) \in B_0$. By Lemma 2, $f(s) \in \arg\min_{f \in B_+} \mathbb{E}_f[s]$. Note that the slope of the south-west boundary of B_+ is either 0, -1, or below -1, where the latter corresponds to the part of the boundary of B_+ that coincides with B_0. The iso-line for $f_1 s_1 + f_2 s_2$ has slope in $[-1, 0]$, and so, the

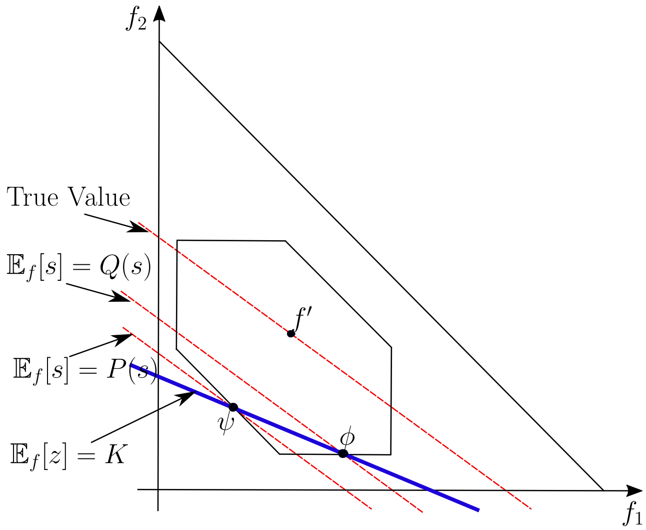


Figure 4: ***Decomposition of mispricing***
Under complete information, security s issued by type f' would be priced at $\mathbb{E}_{f'}[s]$. With private information of the issuer and Knightian uncertainty of the investor, the security is priced at $P(s)$. The mispricing $\mathbb{E}_{f'}[s] - P(s)$ can be decomposed into the mean component, $\mathbb{E}_{f'}[s] - Q(s)$, and the shape component, $Q(s) - P(s)$, where $Q(s)$ is the highest value of security s conditional on the issuer's project having zero NPV (i.e., $f \in B_0$).

minimum of $\mathbb{E}_f[s]$ over $f \in B_+$ is attained on the boundary B_0. Therefore, there is $f \in B_0$ such that $P(s) = \mathbb{E}_f[s]$.

Next, suppose $f(s) \in B_0$, and let us solve the Nature's program

$$\inf_{f \in B} f_1 s_1 + f_2 s_2$$
$$\text{s.t. } f_1 z_1 + f_2 z_2 = K.$$

Using the constraint, we express $f_2 = \frac{K}{z_2} - f_1 \frac{z_1}{z_2}$ and plug this expression for f_2 into the minimized function, we get

$$f_1 s_1 + (K - f_1 z_1)\frac{s_2}{z_2} = K\frac{s_2}{z_2} + f_1\left(s_1 - z_1\frac{s_2}{z_2}\right).$$

The solution has a bang-bang property: $s_1 > z_1\frac{s_2}{z_2}$ implies $f_1 = \phi_1$ and $s_1 < z_1\frac{s_2}{z_2}$ implies $f_1 = \psi_1$. This completes the proof. □

Lemma 4 shows that there is a sharp contrast with the case when the investor is certain that the project has a positive NPV. There, the investor's worst-case justifiable model is $\underline{f}$ for any reasonable security s. In contrast, now the investor's worst-case justifiable model for any reasonable security (i.e., $s \in \mathcal{S}$ at which $P(s) = I$) belongs to a range of distributions with zero NPV (between ψ and ϕ) and depends on the security offered. If the security is concave, then the investor's worst-case justifiable model is distribution ψ, which is the most dispersed distribution among distributions with zero NPV. If the security is convex, then the investor's worst case justifiable model is ϕ, which is the most concentrated among distributions with zero NPV. Thus, the investor's worst-case justifiable model can be either ψ or ϕ, depending on the curvature of the security offered. Respectively, the valuation of the security is either $\mathbb{E}_{\psi}[s]$ or $\mathbb{E}_{\phi}[s]$. The equity stands out as the security with no curvature, and thus, its valuation is the same for all investor's beliefs that are convex combinations of ψ and ϕ.

The following decomposition of the security mispricing will be useful later in providing insights for equilibrium characterizationn. Let

$$Q(s) \equiv \max_{f \in B_0} \mathbb{E}_f[s] \tag{14}$$

be the highest price that the investor is willing to pay for security s if she believes that the model is the most favorable for security s among distributions with zero NPV. We denote by $h(s)$ the solution to (14). By the same argument as in Lemma 4, one can immediately show that

$$h(s) = \begin{cases} \phi & \text{if } s_1 > \frac{z_1}{z_2} s_2, \\ \psi & \text{if } s_1 < \frac{z_1}{z_2} s_2, \\ \alpha\psi + (1-\alpha)\phi, \forall \alpha \in (0,1) & \text{if } s_1 = \frac{z_1}{z_2} s_2, \end{cases}$$

and

$$Q(s) = \mathbb{E}_{h(s)}[s] = \begin{cases} \mathbb{E}_{\phi}[s], & \text{if } s_1 \geq \frac{z_1}{z_2} s_2, \\ \mathbb{E}_{\psi}[s], & \text{if } s_1 \leq \frac{z_1}{z_2} s_2. \end{cases} \tag{15}$$

Note that in determining $Q(s)$ in the expression (15), when the security is concave (convex), the investor believes that the issuer's project has zero NPV, but (unlike in $P(s)$) the cash flow is very concentrated (very dispersed), which results in the higher price $Q(s)$ compared to $P(s)$. Then the security mispricing can be decomposed as follows:

$$\underbrace{\mathbb{E}_f[s] - Q(s)}_{\text{mean component}} + \underbrace{Q(s) - P(s)}_{\text{shape component}}. \tag{16}$$

One can think of the Nature's choice of $f^*(s)$ as a two-stage process. In the first stage, the Nature chooses f to minimize the NPV from the project, but among distributions with the same NPV it chooses the most favorable for security s. We call the change in the security value from this initial transformation the *mean component of mispricing,* and it equals the first term in (16). This component reflects the drop in the security value because of the drop in the NPV.

In the second stage, the Nature chooses f to minimize the payoff from security s among all models with the same NPV chosen in the first stage, which gives us the investor's worst-case justifiable model $f^*(s)$. We call this component the *shape component of the mispricing* and it equals the second term in (16). This component reflects the drop in the security value, as the Nature uses particular details of the security (in this case, curvature) to further lower the value of the security by shifting the probability mass in the distribution of cash flows. Figure 4 illustrates these two stages.

If the investor is confident that the project has a positive NPV, only the mean component is present, as the unique model $\underline{f}$ with the lowest NPV is the worst-case justifiable model for any security. If the investor is uncertain whether the project has positive NPV, there is a range of distributions with zero NPV, and in the first stage, the Nature chooses distribution $h(s)$, the most favorable distribution for security s among all zero-NPV distributions. Hence, in this case, there are both mean and shape component. It is easy to see (e.g., from Figure 4) that the relative contribution of the shape component to mispricing increases with the curvature of the security. In this case, the equity has a distinct property that for equity, the shape component of the mispricing is zero. We will see next that this special feature of equity makes it optimal in many cases.

Characterization of Equilibria We now proceed to the characterization of equilibria.

Proposition 3. *Suppose $\mathbb{E}_f[z] < K$ for some $f \in B$. Then in equilibrium,*

1. *types $f \in D \equiv \{f \in B_+ : \zeta_f > \zeta_\psi\}$ offer a standard risky debt security with face value d ($s(z) = \min\{d, z\}$) given by $\mathbb{E}_\psi[\min\{z, d\}] = I$;*

2. *types $f \in E \equiv \{f \in B_+ : \zeta_\psi > \zeta_f > \zeta_\phi\}$ offer a standard equity security with stake $\frac{I}{K}$ ($s(z) = \frac{I}{K}z$);*

3. *types $f \in C \equiv \{f \in B_+ : \zeta_\phi > \zeta_f\}$ offer a call option with strike price k (($s(z) =$ max $\{z - k, 0\}$)) given by $\mathbb{E}_\phi[\max\{z - k, 0\}] = I$;*

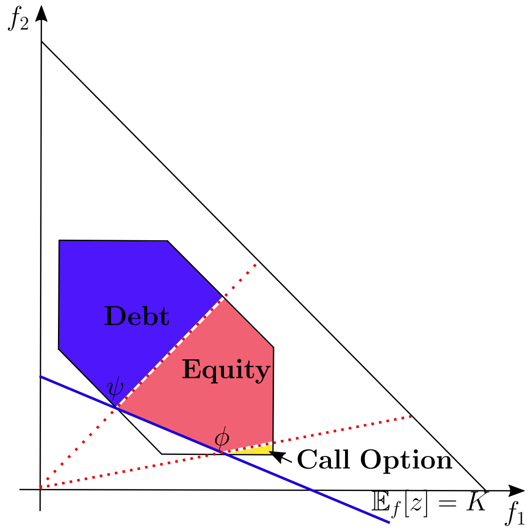


Figure 5: ***Equilibrium when uncertainty is large*** ***(****$\mathbb{E}_f[z] < K$* ***for some*** *$f \in B$* ***)***

4. types $f \in B_-$ do not offer any security.

Proof. Consider the problem of the issuer of type f who chooses s so that $\mathbb{E}_{f^*(s)}[s] = I$ to minimize the mispricing. First, suppose s is concave. Then by Lemma 4, $f^*(s) = \psi$ and the mispricing equals

$$\begin{aligned} \mathbb{E}_f[s] - \mathbb{E}_\psi[s] &= (f_1 - \psi_1)s_1 + (f_2 - \psi_2)s_2 \\ &= (f_1 - \psi_1)s_1 + (\tfrac{f_2}{\psi_2} - 1)(I - \psi_1 s_1) \\ &= \left(\tfrac{1}{\zeta_f} - \tfrac{1}{\zeta_\psi}\right) f_2 s_1 + \left(\tfrac{f_2}{\psi_2} - 1\right) I, \end{aligned}$$

and so, types in D prefer to issue debt to maximize s_1, while types in $B_+ \backslash D$ prefer equity among all concave securities. Similarly, if s is convex, then by Lemma 4, $f^*(s) = \phi$ and the mispricing equals

$$\mathbb{E}_f[s] - \mathbb{E}_\phi[s] = f_2 s_1 + \left(\tfrac{f_2}{\phi_2} - 1\right) I,$$

and so, types in C prefer to issue the call option to minimize s_1, while types in $B_+ \backslash C$ prefer equity among all convex securities. Therefore, types in D prefer to issue debt, types in C prefer to issue call option, and types in E prefer to issue equity.

Let us show that types in B_- prefer not to issue. We show that for all s such that $\mathbb{E}_{f^*(s)}[s] = I$, $\mathbb{E}_f[z - s] - W < 0$ for all $f \in B_-$. Suppose that for some s such that $\mathbb{E}_{f^*(s)}[s] = I$, there exists $f \in B_-$ satisfying $\mathbb{E}_f[z - s] - W \geq 0$. Since $f \in B_-$, $\mathbb{E}_f[z] < K$

and so, $0 \leq \mathbb{E}_f[z-s] - W < K - W - \mathbb{E}_f[s] = I - \mathbb{E}_f[s]$. But then $f \in B(s)$ and $\mathbb{E}_{f^*(s)}[s] - I \leq \mathbb{E}_f[s] - I < 0$, which is a contradiction. □

As before the equilibrium is generically unique up to types on the boundaries of sets D, E, C, which have Lebesgue measure zero and in this sense are not generic. Types on the boundary of D and E (or E and C) are indifferent between debt and equity (resp., equity and call option), or generally, any security that allows them to raise financing in equilibrium. However, as in the previous subsection, all such types necessarily pool on only one security.

Figure 5 illustrates the equilibrium financing characterized in Proposition 3. The novel and most important conclusion of Proposition 3 is that equity stake $\frac{I}{K}$ becomes the optimal source of financing for some types if and only if the investor entertains the possibility that the project has a negative NPV. To see the intuition, it is useful to first turn to the following immediate corollary of Proposition 3 that shows the risky debt can also arise in equilibrium in the case of large uncertainty but under stronger conditions.

Corollary 1. *Suppose that $\mathbb{E}_f[z] < K$ for some $f \in B$. If in equilibrium the investor's worst-case justifiable model, $f^*(s)$, and the distribution of the project's cash flows f are strictly MLRP ordered for any $s \in \mathcal{S}$ such that $P(s) = I$, then the risky debt is the optimal security for the issuer's type f.*

Corollary 1 shows that unlike in the case of small uncertainty, it is no longer sufficient to have $f^*(s^*(f))$ and f strictly MLRP ordered to obtain the optimality of debt, and one needs to check the MLRP ordering for all securities. The reason for this is the following discontinuity in the investor's worst-case justifiable model $f^*(s)$ around the equity security. Recall that for our results, $f^*(s)$ matters only through its effect on $P(s)$ and so, the selection $f^*(s)$ from the solution to the Nature's problem can be arbitrary, but it is convenient for the moment to specify that in (12), $f^*(s) = \phi$ whenever s is equity. Consider type $f \in E$ and suppose he offers some convex security s (e.g., a call option). Then by Lemma 4 such security is priced using $f^*(s) = \phi$. Then the issuer's type f relative to the investor's worst-case justifiable model ϕ assigns a higher probability to state z_2 rather than state z_1. Hence, he has incentives to shift the security payments to relatively less likely state z_1 from relatively more likely state z_2, making the security less convex. This way equity is preferred for such type to any convex security. However, once the issuer offers some strictly concave security s' (e.g., debt), the investor's worst case scenario changes discontinuously from ϕ to ψ. Thus, the security s' is valued at $\mathbb{E}_\psi[s']$ instead of $\mathbb{E}_\phi[s']$. This is precisely the effect of the shape component in mispricing (note that $P(s') = \mathbb{E}_\psi[s']$, while $Q(s') = \mathbb{E}_\phi[s']$), which makes security s' less attractive than equity to type $f \in E$. This is despite the fact that if the investor's worst case scenario were ϕ after the security offer s', the issuer would prefer security s' to the equity financing. Note that by Corollary 1 types $f \in D$, for whom the relative likelihood of cash flow z_2 relative to z_1 is sufficiently high, do prefer to issue the risky debt, as for them the shape component of mispricing is not sufficiently large to outweigh the gain from paying more in the relatively less likely (according to their beliefs) states. However, for types $f \in E$, in equilibrium, the investor's worst-case justifiable model, $f^*(s^*(f))$, and the true cash flow distribution f, are strictly MLRP ordered, but such types do not issue debt, unlike in the small uncertainty case.

Remark 1. *It is interesting to point out that all types in E offer the same equity stake $\frac{I}{K}$. This security has a natural interpretation that the investor gets a share of the cash flow that is proportional to her contribution to the investment K, which is a common profit sharing rule. Our model implies that a variety of project types pool on the same equity contract, which is in contrast with, e.g., the complete information case.*

Optimality of Equity We have shown that in the case of large uncertainty, a range of types finance the project via equity. We next show that equity becomes more prevalent as we make the investment project uniformly worse by increasing the investment cost K or if we increase uncertainty by increasing the size of the uncertainty set (parameter ν). In the next proposition, we use notation B^ν to explicitly state the dependence of B on ν.

Proposition 4. *The following hold:*

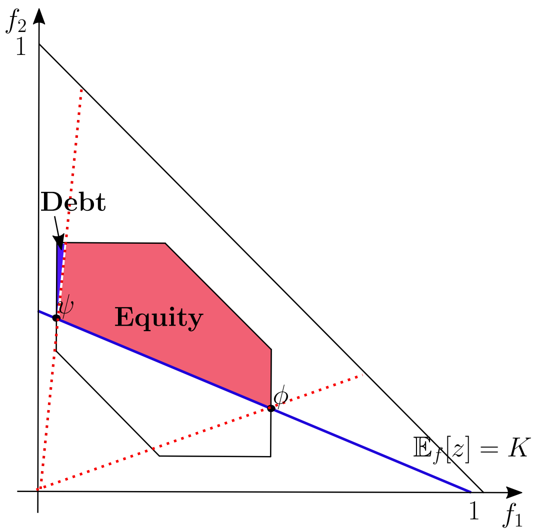
1. *Let $\overline{K} = \max_{f \in B} \mathbb{E}_f[z]$ and suppose B belongs to the interior of $\Delta(Z)$. There exists $\underline{K} < \overline{K}$ such that for all $K \in [\underline{K}, \overline{K}]$, all types of the issuer $f \in B_+$ offer equity and all types $f \in B_-$ forego the investment.*

2. *Let $\overline{\nu}$ be the smallest ν such that $B^\nu = \Delta(Z)$. There exists $\underline{\nu} < \overline{\nu}$ such that for all $\nu \in (\underline{\nu}, \overline{\nu}]$, all types of the issuer $f \in B_+$ offer equity and all types $f \in B_-$ forego the investment.*

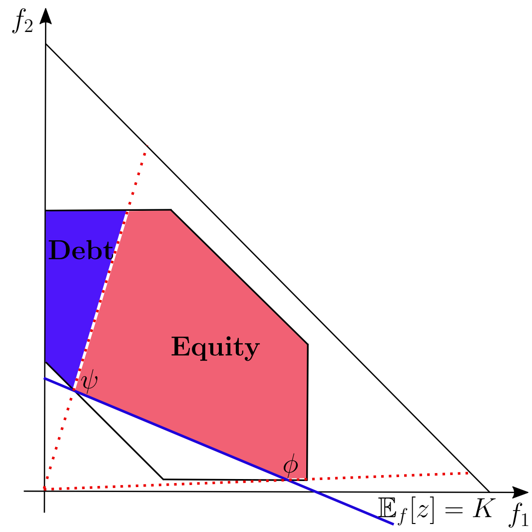
Proof. By Proposition 3, it is sufficient to show that for sufficiently large K or ν, $B_+ = E$.

1) Set $\underline{K} \equiv \max\{g_1 - \nu, 0\} z_1 + \min\{g_1 + \nu, 1\} z_2$, and one can verify that for $K \geq \underline{K}$, ψ belongs to the northern boundary of B and ϕ belongs to the north-eastern or eastern boundary of B, which implies that $B_+ = E$.

2) Set $\underline{\nu}$ to be the smallest ν such that $\{f \in \mathbb{R}^2_+ : \mathbb{E}_f[z] = K\} \subset B^\nu$, which implies that $B^\nu_+ = E$ for all $\nu \geq \underline{\nu}$ and gives the desired result. □



(a) **Worse investment project (higher K)**



(b) **More uncertainty (higher ν)**

Figure 6: ***The effect of an increase in investment level K and uncertainty ν***
In both cases, compared to Figure 5, the region of types that issue equity in equilibrium expands. Set E of types that issue equity is the intersection of B_+ and the cone generated by ψ and ϕ, and this cone expands as K or ν increases. For sufficiently large K and ν, the issuer either raises the funding via equity or foregoes the investment project.

Figures 6*a* and 6*b* illustrate Proposition 3. In Figure 6*a* compared to Figure 5, as K increases, the gap between the two extreme zero-NPV models in B, ψ and ϕ, increases. This way there is a bigger change in the investor's worst-case justifiable model, if the issuer switches from strictly convex to strictly concave securities, and so, fewer types find it optimal to offer call options or debt. As K becomes sufficiently large, all issuer types eventually issue equity in equilibrium. Similarly, in Figure 6*b*, when the uncertainty ν gets larger, the gap between ψ and ϕ increases, which again leads to the expansion of the region of types who issue equity.

Remark 2. *The prediction of our model that the equity becomes more prevalent as uncertainty increases is in sharp contrast with common intuition that debt is the informationally insensitive security and so it should be prevalent in environments with high degree of investors' uncertainty about the issuer's information. In our model, the equity is prevalent in such environments because there is a large change in the investor's worst-case justifiable model when the issuer changes the details of the security (in this case, curvature), which causes additional drop in the security's value. We will show in Section* 6 *that this insight is more general. As we discuss in Section* 7*, our predictions fit existing empirical evidence.*

5 The Case of Private Information about Assets in Place

In this section, we show that when the issuer has private information about assets in place rather than the new project, the equity no longer serves as a positive signal to investors about the cash flow, and the risky debt is optimal under the MLRP ordering of the cash flow distribution and the investor's worst-case justifiable model irrespective of the level of uncertainty.

Assets in Place Model Consider the following modification of the baseline model. (We will use the same notation for objects analogous in the two models). In the baseline model, we assumed that the issuer has an existing resource (or assets in place) of known value W, and he can contribute the value of this resource (e.g., by using assets in the project) to partially cover the investment K in the new project, whose cash flow distribution is the issuer's private information.

Now, suppose that the issuer has assets in place that bring future cash flow z distributed according to f that is privately known by the issuer. If the issuer invests additionally K, the project yields $\hat{z}$ distributed according to $\hat{f}$ given by $\hat{f}_0 = f_0 - \delta, \hat{f}_1 = f_1, \hat{f}_2 = f_2 + \delta$ for some $\delta > 0$. The interpretation is that an investment of K improves the cash flow distribution by moving the probability mass δ from $z = 0$ to $z = z_2$ for any f. The gains from the new project are $\eta \equiv \mathbb{E}_{\hat{f}}[\hat{z}] - \mathbb{E}_f[z] = \delta z_2$. We suppose that the NPV of the new project is positive, i.e., $\eta \geq K$. Thus, this model is the mirror image of the baseline model: The investor and the issuer share common knowledge about what the new project does but the issuer is privately informed about existing assets.

As before, the investor knows that $f \in B$, that is, she considers only distributions in B possible, where B is the total variation distance neighborhood of radius ν around some $g \in \Delta(Z)$. We suppose that δ is sufficiently small so that $\hat{f} \in \Delta(Z)$ for all $\hat{f}$. Since there is a one-to-one mapping between f and corresponding $\hat{f}$, it is more convenient to refer to the distribution of the cash flow after investment, $\hat{f}$, as the issuer's type (instead of f) . Then the issuer's type belongs to the set

$$\hat{B} \equiv B + (0, \delta) = \left\{ \hat{f} \in \Delta(Z) : \hat{f}_0 = f_0 - \delta, \hat{f}_1 = f_1, \hat{f}_2 = f_2 + \delta, \text{ for some } f \in B \right\}.$$

In words, $\hat{B}$ is obtained by shifting every point $f = (f_1, f_2) \in B$ by the vector $(0, \delta)$. Note that $\hat{B}$ is the total variation distance neighborhood of radius ν around $\hat{g} = g + (0, \delta)$.

The issuer offers the security s that pays s_n if $\hat{z} = z_n$, which the investor decides whether to accept or reject. If the issuer fails to raise financing for the new project, he only gets z from the existing project. If the issuer raises financing, then he gets $\hat{z} - s$.

The equilibrium strategies $s^*(\cdot)$ and $\sigma^*(\cdot)$, and the model updating mapping $\hat{B}(\cdot)$ are defined analogously to the baseline model. Specifically, the issuer type $\hat{f} \in \hat{B}$ maximizes his expected utility given σ^*, i.e.,

$$s^*(\hat{f}) \in \arg\max_{s \in \mathcal{S}} \left\{ \sigma^*(s) \mathbb{E}_{\hat{f}}[\hat{z} - s] \right\},$$

and the issuer does not issue any security whenever $\max_{s \in \mathcal{S}: \sigma^*(s)=1} \mathbb{E}_{\hat{f}}[\hat{z} - s] < \mathbb{E}_f[z]$. Note that security s is preferred to security $\tilde{s}$ by the issuer's type $\hat{f}$ if and only if $\mathbb{E}_{\hat{f}}[s] \leq \mathbb{E}_{\hat{f}}[\tilde{s}]$. The issuer prefers to issue security s to not financing the new project if and only if $\mathbb{E}_{\hat{f}}[\hat{z} - s] \geq \mathbb{E}_f[z]$, or after rearranging the terms,

$$\mathbb{E}_{\hat{f}}[s] \leq \mathbb{E}_{\hat{f}}[\hat{z}] - \mathbb{E}_f[z] = \eta. \tag{17}$$

The investor evaluates securities by their worst case scenario: for any $s \in \mathcal{S}$, let

$$P(s) = \min_{\hat{f} \in \hat{B}(s)} \mathbb{E}_{\hat{f}}[s]$$

be the price of security s, and the investor invests if and only if $P(s) \geq K$.

Let $S^* = \cup_{\hat{f} \in \hat{B}} \{s^*(\hat{f})\}$ be the image of $s^*(\cdot)$. Define set $\hat{B}(\cdot)$ as follows. For $s \in S^*$,

$$\hat{B}(s) = cl\{\hat{f} \in \hat{B} : s^*(\hat{f}) = s\},$$

and for $s \notin S^*$,

$$\hat{B}(s) = \left\{ \hat{f} \in \hat{B} : \eta \geq \mathbb{E}_{\hat{f}}[s] \right\},$$

where if the set is empty, $\hat{B}(s) = B$. We denote by $\hat{f}(s)$ the selection from the solution to the minimization problem. (We can fix arbitrary such selection, as the particular selection is not important for our results).

Equilibrium Pricing and Characterization We next show that when the private information is about assets in place, equity is not optimal for any issuer type.

We start with equilibrium pricing of securities. Let $\underline{\hat{f}}$ be the counter-part of type $\underline{f}$, that is, it puts maximal probability $\hat{g} + \nu$ on the lowest state and the minimal probability $\hat{g} - \nu$ on the highest state. Type $\underline{\hat{f}}$ is depicted in Figure 7. By the same argument as in Lemma

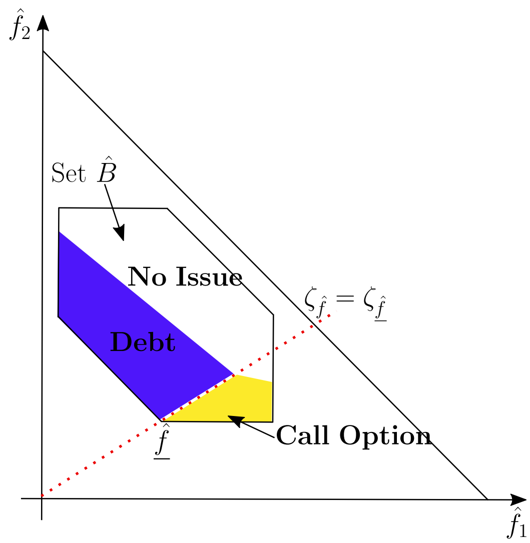


Figure 7: ***Equilibrium in the assets in place model***

3, we can show that for any $s \in \mathcal{S}$, $\mathbb{E}_{\underline{\hat{f}}}[s] = \min_{\hat{f}\in\hat{B}} \mathbb{E}_{\hat{f}}[s]$. The next lemma shows that all securities are prices using the distribution $\underline{\hat{f}}$.

Lemma 5. *In the assets in place model, for any $s \in \mathcal{S}$, $\hat{f}(s) = \underline{\hat{f}}$ and $P(s) = \mathbb{E}_{\underline{\hat{f}}}[s]$.*

In the model with private information about the new project, the interests of the issuer and the investor are partially aligned because the issuer also contributes W to the investment K. In this case, the particular type of security reflects the degree of alignment of the issuer's and investor's interests. In contrast, with uncertainty about assets in place, the interests of the issuer and investor are always opposite: if the issuer manages to convince the investor that the assets in place are of a higher quality, then he can attract the financing more cheaply by paying less in each state. To see this formally, note that the condition (17) implies that type $\underline{\hat{f}}$ prefers to issue s to not issuing any security, whenever at least some other type prefers to issue it. Thus, it is not possible to separate with any security offer from type $\underline{\hat{f}}$ in equilibrium, and in equilibrium, all securities are priced using $\underline{\hat{f}}$. To give a specific example, suppose that some type $\hat{f} \neq \underline{\hat{f}}$ could issue equity stake $\frac{K}{\mathbb{E}_{\hat{f}}[z]}$. Then type $\underline{\hat{f}}$ would gain $K - \frac{K}{\mathbb{E}_{\hat{f}}[z]}\mathbb{E}_{\underline{\hat{f}}}[z] > 0$ compared to his equilibrium payoff from mimicking type $\hat{f}$ and so, investors cannot price this equity stake using the belief $\hat{f}$.

Now, we can characterize the issuer's equilibrium choice of security. Recall that $\zeta_{\hat{f}} = \frac{\hat{f}_2}{\hat{f}_1}$.

Proposition 5. *Let d and k be solutions to $\mathbb{E}_{\underline{\hat{f}}}[\min\{z, d\}] = K$ and $\mathbb{E}_{\underline{\hat{f}}}[\max\{z - k, 0\}] = K$, respectively. In equilibrium, the following hold:*

1. *types* $\hat{f} \in \{\hat{f} \in \hat{B} : \zeta_{\hat{f}} > \zeta_{\underline{\hat{f}}}$ and $\mathbb{E}_{\hat{f}}[\min\{z, d\}]] \leq \eta\}$ *offer risky debt with face value d;*

2. *types* $\hat{f} \in \{\hat{f} \in \hat{B} : \zeta_{\hat{f}} < \zeta_{\underline{\hat{f}}}$ and $\mathbb{E}_{\hat{f}}[\max\{z - k, 0\}] \leq \eta\}$ *offer a call option with strike price k;*

3. *any type* $\hat{f} \in \{\hat{f} \in \hat{B} : \zeta_{\hat{f}} = \zeta_{\underline{\hat{f}}}$ and $\mathbb{E}_{\hat{f}}[\min\{z, d\}]] \leq \eta\}$ *offers any security s satisfying* $\mathbb{E}_{\underline{\hat{f}}}[\tilde{s}^*] = K$*;*

4. *types* $\hat{f}$ *such that* $\mathbb{E}_{\hat{f}}[\min\{z, d\}]] > \eta$ *and* $\mathbb{E}_{\hat{f}}[\max\{z - k, 0\}] > \eta$ *forego the investment.*

By Lemma 27, in equilibrium, the issuer type $\hat{f}$ chooses $s^*(\hat{f})$ to minimizes $\mathbb{E}_{\hat{f}}[s]$ subject to $\mathbb{E}_{\underline{\hat{f}}}[s] = K$, or equivalently, to minimize the mispricing $\mathbb{E}_{\hat{f}}[s] - P(s)$ subject to $\mathbb{E}_{\underline{\hat{f}}}[s] = K$. This problem is identical to the baseline model in the small uncertainty case (analyzed in Subsection 4.2) with types f replaced by $\hat{f}$. Thus, the characterization in Proposition 5 immediately follows from Proposition 2 after we take into account that for certain types their most preferred security that allows them to raise financing is substantially underpriced so that they prefer not to issue any security.

Figure 7 illustrates the equilibrium in the assets in place model. The distinct feature is that irrespective of the level of uncertainty, the equity is never optimal for any type. This follows from the fact that all securities are priced by the investor with belief $\underline{\hat{f}}$ and so, the model with assets in place is similar to the case of small uncertainty of the baseline model.

All types with $\zeta_{\hat{f}} > \zeta_{\underline{\hat{f}}}$ prefer debt to any other security, while all types with $\zeta_{\hat{f}} < \zeta_{\underline{\hat{f}}}$ prefer a call option to any other security. The intuition for the optimality of the risky debt or call option is similar to that in the baseline model in the case of small uncertainty: there is an endogenous heterogeneity of beliefs between the issuer and the investor, and the issuer prefers to maximally shift payments from the security to the states that he considers relatively less likely. Note, however, that in the model with assets in place, some issuer's types prefer not to finance the new project to issuing any security. Those are the types for whom the mispricing is too big and does not cover the gains from investment.

Finally, the next proposition shows that the risky debt is the optimal security whenever the true distribution and the investor's worst-case justifiable model, $\hat{f}$ and $\hat{f}(s)$, are MLRP ordered.

Proposition 6. *In the assets in place model, if in equilibrium investor's worst-case justifiable model* $\hat{f}(s)$ *and the cash flow distribution* $\hat{f}$ *are strictly MLRP ordered, then debt or no financing is optimal for type* $\hat{f}$*.*

6 Generalization to $N > 3$ States

This secion generalizes our main results of the paper to an arbitrary number of states: there are $N+1$ levels of cash flows $0 < z_1 < \cdots < z_N$.

Optimality of Equity We first show that the equity arises in equilibrium when the investment cost K or the uncertainty ν are sufficiently large.

We first generalize two observations from the analysis of the model with three states. The first observation is that when $N = 2$, the set of types that issue a particular security is an intersection of some convex cone and the set B_+. (See Figure 5). The following lemma generalizes this to arbitrary N.

Lemma 6. *The following hold:*

1. *If type f weakly prefers to issue s to $\tilde{s}$, then for any $\gamma > 0$ such that $\gamma f \in \Delta(Z)$, type γf also weakly prefers to issue s to $\tilde{s}$.*

2. *If types f and f' weakly prefer to issue s to $\tilde{s}$, then for any $\alpha \in (0, 1)$, type $f'' = \alpha f + (1-\alpha) f'$ weakly prefers to issue s to $\tilde{s}$.*

3. *For any $s \in \mathcal{S}$ that can be issued on- or off-path, $B(s) = B_+ \cap C(s)$ where $C(s)$ is a convex cone.*

The second observation is that the set of types that issue equity in equilibrium is an intersection of B_+ and the convex cone generated by B_0. (See again Figure 5). Let $\mathcal{E} = B_+ \cap \left(\bigcup_{\psi \in \text{relint}(B_0)} \{\gamma\psi, \gamma \geq 1\} \right)$ be the intersection of B_+ and the cone generated by vectors in relint(B_0), the relative interior of B_0 within the affine hull of B_0. Note that the set $\mathcal{E}$ is non-empty only when B_0 is non-empty, or equivalently, $\mathbb{E}_f[z] \leq K$ for some $f \in B$. The following lemma is the generalization of this observation to arbitrary N.

Lemma 7. *For all $f \in \mathcal{E}$, $s^*(f) = \frac{I}{K} z$.*

The next theorem generalizes Proposition 4 and shows that as the investment cost or the uncertainty increases, equity becomes a dominant source of financing. In the next proposition, we signify by corresponding superscripts the dependence of the set B on ν, and the dependence of B_0, B_+, and $\mathcal{E}$ on ν and K.

Proposition 7. *The following hold:*

1. *Fix ν. There exists $\underline{K} < \overline{K} \equiv \max_{f \in B^\nu} \mathbb{E}_f[z]$ such that for all $K \in [\underline{K}, \overline{K}]$, all types in the interior of $B_+^{\nu,K}$ issue equity in equilibrium.*

2. *Fix K. As $\nu \to \overline{\nu} \equiv \min\{\nu : \Delta(Z) \subseteq B^\nu\}$, $\frac{\Lambda(\mathcal{E}^{\nu,K})}{\Lambda(B_+^{\nu,K})} \to 1$ where Λ denotes the Lebesgue measure.*

Optimality of Debt We next show that the risky debt is optimal under the MLRP ordering of the issuer's and investor's equilibrium beliefs, when the uncertainty about the issuer's project type is relatively small.

We consider the general case with $N+1$ possible values of cash flows. Let $\underline{f} = \arg\min_{f \in B} \mathbb{E}_f[z]$.

Lemma 8. *There exists a unique distribution $\underline{f} \in B$ such that for any $s \in \mathcal{S}$, $\underline{f}$ is the unique solution to $\min_{f \in B} \mathbb{E}_f[s]$. Therefore, if $\mathbb{E}_f[z] \geq K$ for all $f \in B$, then $f^*(s) = \underline{f}$ for all $s \in \mathcal{S}$.*

As before, we say that two distributions f and f' are strictly MLRP-ordered if $\frac{f_n}{f'_n}$ is either strictly increasing or strictly decreasing in n. By the same argument as in the three-states model, we get the following generalization of Proposition 2.

Proposition 8. *Suppose that $\mathbb{E}_f[z] \geq K$ for all $f \in B$. If in equilibrium the investor's worst-case justifiable model $\underline{f}$ and the true cash flow distribution f are strictly MLRP ordered, then the issuer with type f finances the project with debt.*

Assets in Place Finally, we generalize the assets in place model analyzed in Section 5 to $N+1$ states.

As in the three-state assets in place model, the issuer has private information about the distribution of cash flows from the assets in place, $f \in B$. If the investment K is made, the cash flow distribution becomes $\hat{f}$. We suppose that there are real numbers $\{\delta_n\}_{n=0}^N$ such that for any $f \in B$ and $n \in \{0, \ldots, N\}$, $\hat{f}_n - f_n = \delta_n$. We suppose that δ_n are common knowledge and $\sum_{n=0}^N \delta_n > 0$. That is, while the cash flow from the assets in place is not common knowledge, it is common knowledge how the investment alters cash flows. The rest of the formulation is as in Section 5. One can verify that the proof of Lemma 5 does not rely on the particular transformation of f into $\hat{f}$, and so, the Nature's choice is $\underline{\hat{f}}$ irrespective of whether there are negative NPV f in B or not. Moreover, the logic of the optimality of debt in the assets in place model in Section 5 goes through, and we get that under the strict MLRP ordering of $\underline{\hat{f}}$ and $\hat{f}$, debt or no financing is optimal in the general model.

7 Discussion

In this section, we relate our results to the empirical tests of the pecking order theory, compare our model of financing under asymmetric information and Knightian uncertainty to modelf of financing under asymmetric information and Bayesian uncertainty and to models of moral hazard.

Empirical Implications This subsection connects our predictions to recent empirical tests of the pecking order theory. Let us first summarize main empirical predictions of our model:

1. When the uncertainty about the new project is small, debt is the optimal under the MLRP ordering of beliefs (Proposition 2).
2. When the uncertainty about the new project is large, equity is the optimal security (Proposition 4).
3. When the private information is about assets in place, debt is optimal under the MLRP ordering of beliefs (Proposition 5).

The classical pecking-order theory proposed by Myers and Majluf (1984) states that information asymmetry leads to the issuer's preference for financing through raising debt rather than issuing equity. There is at best mixed evidence about the validity of the pecking order theory. Shyam-Sunder and Myers (1999) show that for a sample of mature firms, there is a strong relation between the financial deficit and net debt issuance. Based on this evidence, they conclude that the data support the pecking order theory. Frank and Goyal (2003) show that for small, high-growth firms, this relationship is no longer present. They reason that for such firms the information asymmetries should be a significant concern, and thus, one should expect the support for the pecking order theory to be more pronounced. Based on the fact that they find the opposite, they reject the pecking order theory.

This evidence, however, is in line with our theory, which stresses the nature of the private information for the ordering of securities. For mature firms, the value comes mostly from assets in place and our theory is in accord with the pecking order theory. For young, high-growth firms, the private information is more likely to be about the new project, and equity is optimal particularly when the uncertainty is large. Thus, results by Shyam-Sunder and Myers (1999) and Frank and Goyal (2003) are in line with our predictions.

Relation to Bayesian Signaling In this subsection, we compare our model of security design under Knightian uncertainty to models of security design under Bayesian uncertainty.

Formally, our model belongs to a general class of signalling games with multidimensional types (in our case, distributions of cash flow) and multidimensional signals (in our case, mappings from the future cash flow into the security payment). In general, this problem with Bayesian receivers is considered to be very hard, and the characterization of the equilibrium set remains an open question. The existing theoretical work on multidimensional signalling (see Quinzii and Rochet (1985), Engers (1987)) provides sufficient conditions for separating equilibria. The existing applied work, which includes models of signalling with securities, makes strong assumptions on the ordering of types to essentially reduce the analysis to single-dimensional types. The closest to our paper in the Bayesian signalling literature is the classical paper by Nachman and Noe (1994), and we use it next to illustrate the difference of our robust approach.[17]

There are two key differences of our model. First, is the robust approach to security valuation by the investor. Second, Nachman and Noe (1994) impose a strong ordering on the issuer's types, which allows them to essentially reduce the problem to one-dimensional signalling. In contrast, we put very little structure on the possible distributions of cash flows, but only require that the belong to set B.

The robust approach allows us to provide the sharp characterization of equilibria of the cash raising game under weaker assumptions about the possible distributions of cash flow. We will next demonstrate that relaxing the assumption about the structure of the issuer's private information allows us to uncover novel features in financing under asymmetric information, and in particular, to show that equity may be optimal when the uncertainty is large. To see this, let us first consider the model of Nachman and Noe (1994) with three states. In their model, issuer's type θ belongs to a finite set $\theta \in \Theta$, and each type is associated with the distribution $f(\theta)$ of future cash flows from the new project. Nachman and Noe (1994) assume that types in Θ are ordered by the strict conditional stochastic dominance (SCSD) ordering. With three states, SCSD ordering of types implies that for all $\theta, \theta' \in \Theta$ such that $\theta < \theta'$,

$$\begin{cases} \frac{f_1(\theta')}{f_1(\theta)} < \frac{f_2(\theta')}{f_2(\theta)}, \\ f_1(\theta) + f_2(\theta) < f_1(\theta') + f_2(\theta'). \end{cases}$$

Figure 8 provides examples of SCSD-ordered set Θ, and in particular, demonstrates that types in Θ are non-generic in our model. Nachman and Noe (1994) additionally impose

[17] In fact, an attentive reader may notice that in the title we intentionally mirrored their title.

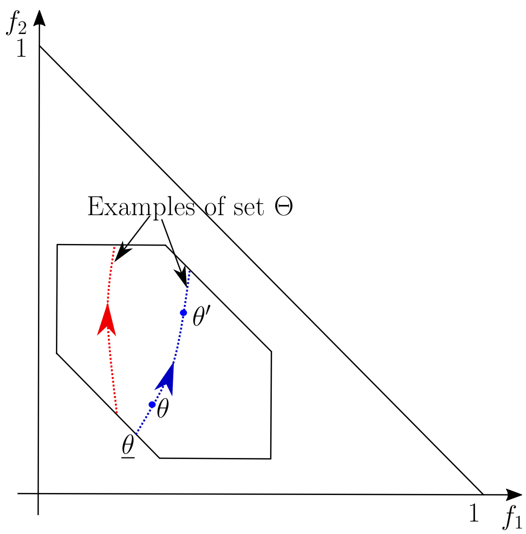


Figure 8: ***Examples of set*** Θ ***in*** *Nachman and Noe (1994)*
Arrows indicate the direction of the increase of types.

the D1 refinement and assume that all types have positive NPV. Their main result is that all types in Θ pool on the risky debt. By following the same argument as in the case of small uncertainty in Subsection 4.1, we can show that the same result also obtains in our model, when types are SCSD-ordered.[18] The important insight of our paper is that without strong restrictions on the set of issuer's types, equity naturally arises as an optimal source of financing, and it is closely related to the level of the investor's uncertainty.

Relation to Models of Robust Contracting under Moral Hazard Our theory generates standard risky debt and standard outside equity as equilibrium securities based on asymmetric information between the issuer and the investor. An alternative explanation for these securities comes from models of moral hazard. In a classic paper, Innes (1990) shows that selling debt is the optimal way to finance a project when the entrepreneur faces a moral hazard problem under risk neutrality and limited liability. In a similar moral hazard setting but assuming that the principal faces nonquantifiable uncertainty and requires robustness, Carroll (2015) shows that the optimal contract is linear. Related ideas that linear contracts, in particular, equity, are robust contracts to moral hazard problems also appear in Holmstrom and Milgrom (1987), Admati and Pfleiderer (1994), and Ravid and Spiegel (1997).

[18]The sketch of the argument is as follows. First, let $\underline{\theta}$ be the lowest type according to the SCSD ordering. (See Figure 8 for the illustration). Similarly to Lemma 3, all securities are priced using $f(\underline{\theta})$. Note that with three states the MLRP and the SCSD orderings coincide. By the argument analogous to Proposition 2, all types issue debt in equilibrium.

Thus, our theory shares a common prediction with moral hazard models that one should observe financing via equity when Knightian uncertainty is very high and via debt when it is very low.

There are two conceptual differences in implications of our model based on asymmetric information from the moral hazard model of Innes (1990) and Carroll (2015). The first conceptual difference concerns the predictions of the model when Knightian uncertainty is present (unlike Innes (1990)) but not extreme (unlike Carroll (2015)). The moral hazard problem with these features is analyzed recently by Antic (2015), and the optimal contract in this case is neither standard risky debt nor standard outside equity. In contrast, standard risky debt and standard outside equity arise in non-extreme versions of the model based on asymmetric information - as we saw, standard outside equity is issued by some types whenever the investor contemplates that the project has a negative NPV, and standard risky debt arises is issued by some types whenever uncertainty is not too high. That is, the specialness of these securities in our model does not rely on Knightian uncertainty being extremely high or absent.

The second conceptual difference concerns the importance of the nature of private information. Our model implies a big difference between the case when private information of the issuer and uncertainty of the investor concern cash flows of the new project with the case when they concern existing assets in place. In particular, standard outside equity arises in equilibrium in the former case, but never in the latter case, no matter how high Knightian uncertainty is. In contrast, moral hazard models do not imply this difference.

8 Robustness

This section shows that our results are robust to what information the investor learns from offers, the investor's valuation technique, and the specification of the uncertainty set.

8.1 Alternative Definitions of the Set of Justifiable Models

In our model, the notion of the set of justifiable models captures the idea that the investor tries to learn as much as possible even from offers that lie out of equilibrium path. We next identify the element of investor's learning that is crucial for our results.

Let us consider two alternative specifications of the model updating mapping. First, suppose $B(s) = B$ for any off-path security s, that is, the investor does not learn from off-path offers. Then we generally can sustain a variety of equilibria in this case. To see this, consider the case when $\mathbb{E}_{\underline{f}}[z] < I$. Consider some security $\tilde{s}$ such that $\mathbb{E}_f[\tilde{s}] = I$ for some $f \in B$. Then there is an equilibrium, in which all types f such that $\mathbb{E}_f[z - \tilde{s}] \geq W$ issue $\tilde{s}$, while the rest of types do not raise financing. Indeed, any off-path offer s will be rejected, as $\min_f \mathbb{E}_f[s] = \mathbb{E}_{\underline{f}}[z] < I$. Thus, no security other than $\tilde{s}$ could be issued, and security $\tilde{s}$ is issued only by types with $\mathbb{E}_f[z - \tilde{s}] \geq W$. This implies that the investor's learning from both on- and off-path security offers is important for our generic uniqueness result in Theorem 1.

In the second specification, suppose that

$$B(s) = \{f \in B : \mathbb{E}_f[z - s] \geq W\} \tag{18}$$

whenever this set is non-empty, and $B(s) = B$ otherwise. The intuition for this criterion is that offering security s if $\mathbb{E}_f[z - s] < W$ is dominated by not raising financing if there is even a slightest chance that such a security will be accepted. Thus, the investor discards all types for whom this is the case. One can verify that our analysis goes through under this specification of the model updating rule. The fact that our results hold for this alternative specification of model updating mapping is closely related to the conclusion of Lemma 2. Recall that Lemma 2 implies that while in general the investor learns certaint information from security offers, the only information that is relevant for her valuation of security offers is whether the project has positive or negative NPV. This, however, is not obvious a priori, as one might expect that the more refined information that the issuer signals through security offers could improve the valuation of securities and result in further separation in equilibrium, and one of the implications from Lemma 2 is that this is not the case.

8.2 Separation of Uncertainty and Uncertainty Aversion

This subsection shows that our results are robust to an alternative robust valuation method used by investors.

In venture financing, investors often aim to limit losses in the worst-case scenario, while ensuring there is a significant upside in the best-case scenario (the "catch a unicorn"). The Hurwitz criterion captures this valuation method. Specifically, for some fixed $\omega \in (0, 1]$, the investor values security s at

$$P^{\omega}(s) = \omega \min_{f \in B(s)} \mathbb{E}_f[s] + (1 - \omega) \max_{f \in B(s)} \mathbb{E}_f[s] \tag{19}$$

instead of (3). This valuation arises when the investor lacks confidence to assign probability

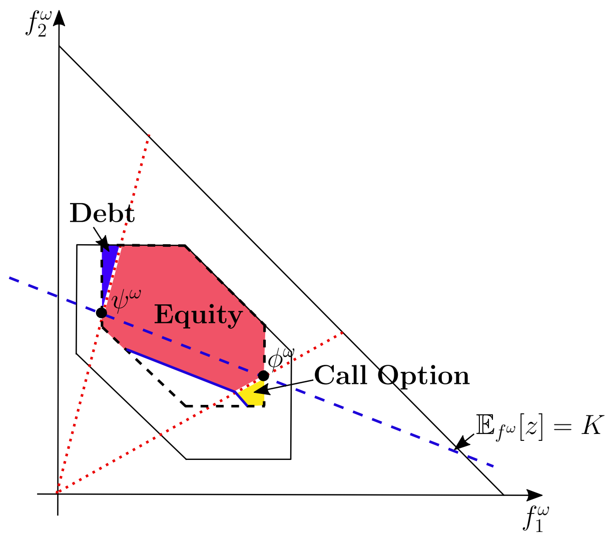


Figure 9: ***Illustration of Equilibria when Securities are Valued at*** P^ω
Pooling regions are described in terms of types f^ω *inside the set* B^ω *(the dashed hexagon), which is a contracted version of* B.

to all possible scenarios or lacks data to estimate these probabilities. Instead, she simplifies the problem and focuses on the weighted average of worst- and best-case scenarios. When $\omega = 1$, (19) reduces to our baseline model, while when $\omega \to 0$, the investor becomes non-prudent and only focuses on the best-case scenario. Ghirardato et al. (2004) provide axiomatic foundations for the Hurwitz criterion.

We next extend our analysis to this more general model of the valuation of securities by investors. For tractability, we deviate from the base model and follow the previous section by assuming that $B(s) = \{f \in B : \mathbb{E}_f[z - s] \geq W\}$.

By the same logic as in Lemma 3, we can show that if $\overline{f}$ is the unique distribution that maximizes $\mathbb{E}_f[z]$ over $f \in B$, then $\max_{f \in B} \mathbb{E}_f[s] = \mathbb{E}_{\overline{f}}[s]$ for all $s \in \mathcal{S}$. Thus, given that $B(\cdot)$ is given by 18, $P^\omega(s)$ in (19) can be rewritten as

$$P^\omega(s) = \omega \min_{f \in B(s)} \mathbb{E}_f[s] + (1 - \omega)\mathbb{E}_{\overline{f}}[s]$$
$$= \min_{f \in B(s)} \mathbb{E}_{\omega f + (1-\omega)\overline{f}}[s].$$

Define $B^\omega = \{f^\omega = \omega f + (1 - \omega)\overline{f}, f \in B\}$, and map any $f \in B$ into corresponding distribution $f^\omega = \omega f + (1 - \omega)\overline{f} \in B^\omega$. (See Figure 9). We can prove that the counter-parts of Lemmas 1 and 2 hold (Lemma 12 in the Online Appendix). In particular, it is without loss for the equilibrium analysis to restrict attention to securities such that $P^\omega(s) = I$, and for all such securities,

$$P^\omega(s) = \min_{f^\omega \in B_+^\omega} \mathbb{E}_{f^\omega}[s],$$

where $B_+^\omega = \{f^\omega \in B^\omega : \mathbb{E}_{f^\omega}[z] \geq K\}$ is the counter-part of B_+ in the baseline model. By the same argument as in the baseline model, the issuer of type f chooses among securities $s \in \mathcal{S}$ such that $P^\omega(s) = I$ the one that minimizes $\mathbb{E}_f[s]$. Since for any securities s and s' in $\mathcal{S}$, $\mathbb{E}_f[z-s] \geq \mathbb{E}_f[z-s']$ if and only if $\mathbb{E}_{f^\omega}[z-s] \geq \mathbb{E}_{f^\omega}[z-s']$, this minimization is equivalent to minimizing $\mathbb{E}_{f^\omega}[s]$ subject to $P^\omega(s) = I$.

The solution to this problem is similar to the baseline model, and we describe it in the case of large uncertainty. Denote

$$\psi^\omega = \arg\min_{f^\omega \in B^\omega} \{f_1^\omega \text{ s.t. } f_1^\omega z_1 + f_2^\omega z_2 = K\},$$
$$\phi^\omega = \arg\max_{f^\omega \in B^\omega} \{f_1^\omega \text{ s.t. } f_1^\omega z_1 + f_2^\omega z_2 = K\},$$

the counter-parts of ψ and ϕ in the baseline model. Then for types f^ω with $\zeta_{f^\omega} > \zeta_{\psi^\omega}$ the most preferred security is risky debt d such that $\mathbb{E}_{\psi^\omega}[\min\{d, z\}] = I$, types f^ω with $\zeta_{f^\omega} \in (\zeta_{\phi^\omega}, \zeta_{\psi^\omega})$ the most preferred security is equity $\frac{I}{K}z$, and f^ω with $\zeta_{f^\omega} < \zeta_{\phi^\omega}$ the most preferred security is call option with strict k such that $\mathbb{E}_{\phi^\omega}[\min\{z-k, 0\}] = I$. Whether they issue these securities in equilibrium or prefer not to raise financing depends on whether their payoff exceeds W or not, i.e., for appropriate security whether $\mathbb{E}_f[z-s] \geq W$ or not. This implies that a larger set of types compared to B_+^ω raises financing in equilibrium.

We conclude that equilibria of the version of the model where the investor's valuation is given by (3) are equivalent to equilibria in the model where B is replaced by B^ω, types f are replaced by f^ω, with the adjustment that types f^ω with $\mathbb{E}_{\frac{f^\omega-(1-\omega)\bar{f}}{\omega}}[z-s] \geq W$ issue securities in equilibrium (rather than only types with $\mathbb{E}_{f^\omega}[z-s] \geq W$). In such a model, the set B^ω is a subset of B and so, debt would be more prevalent compared to equity. Thus, the change in the valuation method from baseline to (3) is similar to the reduction in the investor's uncertianty about possible distributions of cash flow.

8.3 Specification of the Uncertainty Set

In the base model, we used the total variation distance as a measure of distance between two probability distributions. In this subsection, we show that our main results are robust to the specification of set B. We demonstrate the robustness of our results using the Euclidean model specification, as it is another most commonly used metric in economics and finance, however, as we discuss in Remark 3, our analysis can be applied more generally to any convex B with the non-empty interior.

Formally, suppose that in the baseline model with three states, the set B is a Euclidean ball of radius ν around $\mathbf{f}$, i.e.,

$$B \equiv \left\{ f \in \Delta(Z) : \sum_{n=0}^{2} (f_n - \mathbf{f}_n)^2 \leq \nu^2 \right\}.$$

The examples of such a set are depicted below in Figures 10a and 10b.

The first observation is that that proofs of the results about the equilibrium valuation of securities in Section 3 are valid irrespective of the specification of set B. Thus, as before, for any $s \in \mathcal{S}$, the equilibrium pricing $P(s)$ of securities is given by the expression (7).

The Case of Small Uncertainty Let us start with the case of small uncertainty, i.e., $\mathbb{E}_f[z] \geq K$ for all $f \in B$. We can find $P(s)$ in this case as follows. We can represent any security $s \in \mathcal{S}$ by two parameters: $\alpha_s = \frac{s_1}{s_2} \in [0,1]$ and $\beta_s = \frac{s_2}{z_2} \in [0,1]$.[19] Parameter α_s reflects the curvature of the security s, and β_s is a scaling parameter. For $\alpha \in [0,1]$, let

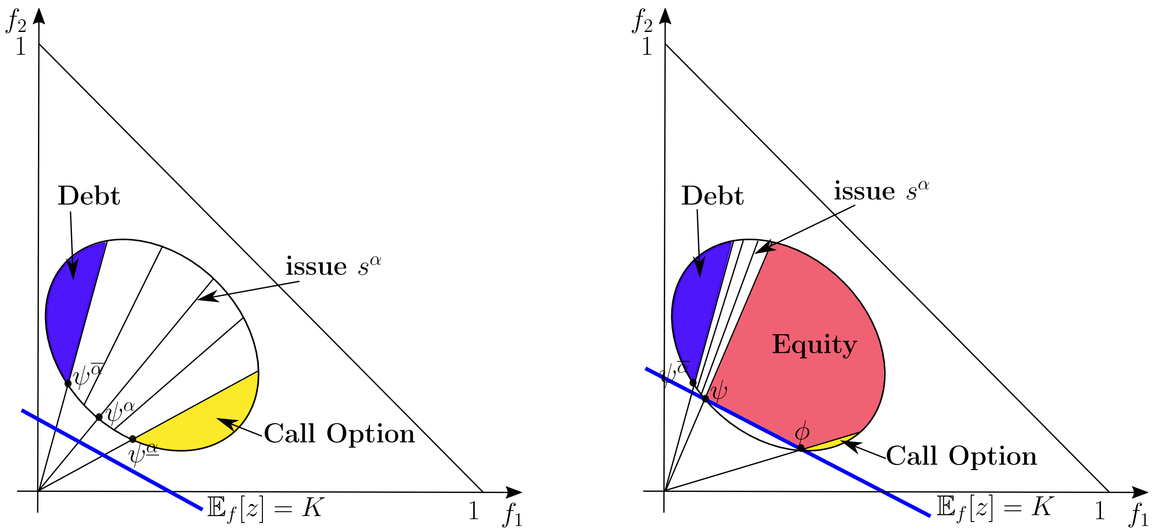
$$\psi^{\alpha} \equiv \arg\min_{f \in B} \{\alpha f_1 + f_2\}. \tag{20}$$

Graphically, the problem (20) boils down to finding the point, at which the line $\{(f_1, f_2) : \alpha(f_1 - \psi_1^{\alpha}) + (f_2 - \psi_2^{\alpha}) = 0\}$ is tangent to the south-western boundary of B. (See an example of ψ^{α} in Figure 10a). Since B is a strictly convex set and $\alpha f_1 + f_2$ is linear, there is a unique such point ψ^{α}. Since ψ^{α} solves (7) for any s with $\alpha_s = \alpha$, we get the following lemma describing the equilibrium pricing in the case of small uncertainty:

Lemma 9. *Suppose $\mathbb{E}_f[z] \geq K$ for all $f \in B$. Then for any $\alpha \in [0,1]$ and any security $s \in \mathcal{S}$ such that $\alpha_s = \alpha$, we have $P(s) = \mathbb{E}_{\psi^{\alpha}}[s]$.*

For any $\alpha \in [0,1]$, let s^{α} be the security such that $\mathbb{E}_{\psi^{\alpha}}[s^{\alpha}] = I$ and $\frac{s_1^{\alpha}}{s_2^{\alpha}} = \alpha$, whenever such a security exists. We show in the appendix that there is a maximal set $[\underline{\alpha}, \overline{\alpha}]$ such that s^{α} is well-defined for all $\alpha \in [\underline{\alpha}, \overline{\alpha}]$. Then by the same argument as in Lemma (1), only securities in $\{s^{\alpha}, \alpha \in [\underline{\alpha}, \overline{\alpha}]\}$ are issued in equilibrium, which are priced in equilibrium at $P(s^{\alpha}) = \mathbb{E}_{\psi^{\alpha}}[s^{\alpha}]$.

[19] Note that there is a one-to-one correspondence between s and (α_s, β_s) given by $s_1 = \alpha_s \beta_s z_2$ and $s_2 = \beta_s z_2$.



(a) **Small uncertainty:** $\mathbb{E}_f[z] \geq K$ **for all** $f \in B$

(b) **Large uncertainty:** $\mathbb{E}_f[z] < K$ **for some** $f \in B$

Figure 10: ***Equilibrium in the Euclidean model***

We can follow the same line of argument as in the analysis of the baseline model to characterize the equilibrium securitiy offers. Each issuer's type f chooses the security $s^*(f)$ that minimizes the mispricing $\mathbb{E}_f[s] - \mathbb{E}_{f(s)}[s]$ subject to $\mathbb{E}_{f(s)}[s] = I$, or equivalently, solves

$$\alpha \in \arg \min_{\alpha \in [\underline{\alpha}, \overline{\alpha}]} \left\{ \mathbb{E}_f[s] - \mathbb{E}_{\psi^\alpha}[s^\alpha] \right\}. \tag{21}$$

The solution to this problem gives us the following characterization of equilibria in the case of small uncertainty (recall that we defined $\zeta_f = \frac{f_2}{f_1}$):

Proposition 9. *Consider the Euclidean model and suppose $\mathbb{E}_f[z] \geq K$ for all $f \in B$. Then in any equilibrium of the signaling game, the following hold:*

1. *For any $\alpha \in (\underline{\alpha}, \overline{\alpha})$, the issuer type $f \in \{f \in B : \zeta_f = \zeta_{\psi^\alpha}\}$ offers security s^α, i.e., $s^*(f) = s^\alpha$.*

2. *Issuer types $f \in \{f \in B : \zeta_f \geq \zeta_{\psi^{\overline{\alpha}}}\}$ offer security $s^{\overline{\alpha}}$, i.e., $s^*(f) = s^{\overline{\alpha}}$. Moreover, security $s^{\overline{\alpha}}$ is the risky debt d such that $\mathbb{E}_{\psi^{\overline{\alpha}}}[\min\{z, d\}] = I$.*

3. *Issuer types $f \in \{f \in B : \zeta_f \leq \zeta_{\psi^{\underline{\alpha}}}\}$ offer security $s^{\underline{\alpha}}$, i.e., $s^*(f) = s^{\underline{\alpha}}$. Moreover, security $s^{\underline{\alpha}}$ is the call option with the strike price k such that $\mathbb{E}_{\psi^{\underline{\alpha}}}[\min\{z - k, 0\}] = I$.*

Figure 10*a* depicts a typical equilibrium in the case of small uncertainty. Similarly to the baseline model, there are two regions of types with sufficiently large and sufficiently small ζ_f

that pool on the risky debt $s^{\overline{\alpha}}$ or call option $s^{\underline{\alpha}}$, respectively. However, unlike the baseline model, in the Euclidean model, there is a range of types with $\zeta_f \in \left(\zeta_{\psi^{\underline{\alpha}}}, \zeta_{\psi^{\overline{\alpha}}}\right)$ that partially separate. Specifically, all issuer types with $\zeta_f = \zeta_{\psi^\alpha}$ offer security s^α. Graphically, these are the types that lie on the part of the line with direction ψ^α inside the set B. The feature of the Euclidean model that allows for such a separation is that the slope of the south-eastern boundary of B continuously changes from $-\infty$ to 0, which is in contrast to the baseline model, where it can be only $-\infty$, -1, or 0.

Observe that the strict MLRP ordering of the issuer's belief $f(s^*(f))$ and the investor's belief f holds only when $\zeta_f > \zeta_{\psi^{\overline{\alpha}}}$, and the issuer offers debt in this case. Thus, Proposition **??** holds in the Euclidean model implying that the risky debt is optimal under the strict MLRP ordering of players' equilibrium beliefs.

The Case of Large Uncertainty Let us turn now to the case of large uncertainty. We define ϕ and ψ as in expressions (10) and (11). By the same logic as in Section 4.2, the issuer's types with $\zeta_f \in (\zeta_\phi, \zeta_\psi)$ offer equity. The rest of the types issue the same securities that they issued in the case of small uncertainty whenever it is possible, otherwise, they pool on the risky debt or call option. The following proposition characterizes the equilibria in the case of large uncertainty:

Proposition 10. *Consider the Euclidean model and suppose $\mathbb{E}_f[z] \geq K$ for all $f \in B$. Then in any equilibrium of the signaling game, the following hold:*

1. *Issuer types f with $\zeta_f \in (\zeta_\phi, \zeta_\psi)$ issue equity $s = \frac{I}{K}z$.*
2. *For issuer types f with $\zeta_f > \zeta_\psi$, it holds*
   (a) *if $\zeta_\psi \geq \zeta_{\psi^{\overline{\alpha}}}$, then types $f \in \{f \in B : \zeta_f > \zeta_\psi\}$ issue risky debt d such that $\mathbb{E}_\psi[\min\{z, d\}] = I$;*
   (b) *if $\zeta_\psi < \zeta_{\psi^{\overline{\alpha}}}$, then for any α such that $\zeta_{\psi^\alpha} \in (\zeta_\psi, \zeta_{\psi^{\overline{\alpha}}})$, types $f \in \{f \in B : \zeta_f = \zeta_{\psi^\alpha}\}$ issue s^α, and types $f \in \{f \in B : \zeta_f > \zeta_{\psi^{\overline{\alpha}}}\}$ issue risky debt d such that $\mathbb{E}_{\psi^{\overline{\alpha}}}[\min\{z, d\}] = I$;*
3. *For issuer types f with $\zeta_f < \zeta_\phi$, it holds*
   (a) *if $\zeta_\phi \leq \zeta_{\psi^{\underline{\alpha}}}$, then types $f \in \{f \in B : \zeta_f < \zeta_\phi\}$ issue call option with strike k such that $\mathbb{E}_\phi[\max\{z - k, 0\}] = I$;*

(b) if $\zeta_\phi > \zeta_{\psi^{\underline{\alpha}}}$, then for any α such that $\zeta_{\psi^\alpha} \in (\zeta_{\psi^{\underline{\alpha}}}, \zeta_\phi)$, types $f \in \{f \in B : \zeta_f = \zeta_{\psi^\alpha}\}$ issue s^α, and types $f \in \{f \in B : \zeta_f < \zeta_{\psi^{\underline{\alpha}}}\}$ issue call option with strike k such that $\mathbb{E}_{\zeta_{\psi^{\underline{\alpha}}}}[\max\{z-k, 0\}] = I$.

The typical equilibrium is depicted in Figure 10*b*. The equilibrium structure is quite similar to that in the baseline model with the difference that there is an additional separation of certain types that offer securities s^α. (Compare Figures 5 and 10*b*). Moreover, similarly to the baseline model, as K or ν increase, the equity still becomes the dominant source of financing, as the cone $\{f \in B : \zeta_f \in (\zeta_\phi, \zeta_\psi)\}$, in which it is issued, expands.

Remark 3. *In this section, we carried the analysis for $N = 2$ and the Euclidean specification of set B. One can verify that the results for general N in Section* 6 *also hold for the Euclidean specification. Concerning more general specifications of B, our analysis reveals that the slope of the south-western boundary of B determines whether the issuer offers securities different from debt, equity, and call option to partially separate. Our analysis can be carried out for general convex B with non-empty interior. While the shape of B affects the details of equilibrium strategies, the general message that 1) equity becomes dominant as K or ν increase, and 2) the optimality of debt in the case of small uncertainty under the strict MLRP ordering of the players' valuation models are robust to the specification of set B.*

9 Conclusion

The objective of the paper is to analyze the classical problem of optimal financing under asymmetric information when the investor is uncertain about the cash flow distribution in the Knightian, rather than Bayesian, sense. The investor only knows that the cash flow distribution is within a neighborhood of a certain base distribution and demands robustness evaluating any security by the worst-case distribution at which the investor could justify the issuer offering that security.

Our analysis generates two insights. First, the model rationalizes two most common financial contracts, standard outside equity and standard risky debt, as usual equilibrium outcomes. Outside equity is special because it serves as a very credible signal that the project is "good enough", because the investor and the issuer both hold security with the same shape. In contrast, any other security only sends the message that the residual security (i.e., the one kept by the issuer) is "good enough". Standard risky debt is special, because it gives the lowest possible sensitivity of the payoff to the cash flow, which is valued by a cautious investor. While there are many models providing foundations for standard risky debt, rationalizing outside equity has been more difficult, and it usually relies on very different arguments than models rationalizing debt. For example, Fluck (1998) and Myers (2000) rationalize outside equity as the optimal relational contract between insiders and outside investors. In our model, both securities are rationalized with one simple market imperfection, private information of the issuer, provided that the investor faces uncertainty in the Knightian sense.

The second insight is to relate the equilibrium security (risky debt or outside equity) to economic environment. In our view, the most interesting implication is that outside equity arises in equilibrium only if the issuer's private information concerns a new project. In contrast, if the issuer's private information is about assets in place, then outside equity never arises in equilibrium, because it is a credible signal that the project is good enough, but not that the assets in place are good enough. Another implication is that when private information concerns a new project, outside equity arises in equilibrium when uncertainty is high, while risky debt arises when it is low. While refined empirical tests need to be conducted, at first glance these implications appear to be in line with the existing empirical evidence on the validity of the pecking order theory.

A Appendix

Proof of Proposition 1. Recall that we defined $S^*(f)$ in the text as the solution to program 6, or equivalently, 8, and by the argument in the main text, $s^*(f) \in S^*(f)$.

First, observe that any type $f \in B_-$ does not make any offer in equilibrium. Indeed, if offer s is made in equilibrium by type $f \in B_-$, then $\mathbb{E}_f[z-s] \geq W$, but then $P(s) \leq \mathbb{E}_f[s] \leq \mathbb{E}_f[z] - W < K - W = I$, which contradicts the fact that in equilibrium, the investor accepts on-path offers. Thus, in what follows we focus on types $f \in B_+$.

Observe that if for some type $f \in B_0$, $S^*(f)$ is a singleton, then for any $\gamma \geq 1$ and type $f' = \gamma f \in B$, $S^*(f')$ is a singleton. (This follows from the fact that for any s and s' in $\mathcal{S}$, $\mathbb{E}_f[s] \geq (>) \mathbb{E}_f[s']$ if and only if $\mathbb{E}_{f'}[s] \geq (>) \mathbb{E}_{f'}[s']$). Since set B_0 is $N-1$ dimensional,[20] it is sufficient to show the following claim:

Claim 1. *The set of types in B_0, for which $S^*(f)$ is not a singleton, is of dimension less than or equal $N-2$.*

Claim 1 implies that the set of types in B_+, for which $S^*(f)$ is a singleton, has dimension N and gives the desired conclusion.

Let us now show Claim 1. Note that B_+ is a convex polyhedron and so, it has a finite number of extreme points $G \equiv \{g^1, \ldots, g^I\} \subset B_+$. Since the minimum of a linear function on a convex

[20]Recall the dimensionality of $\Delta(Z)$ is N.

polyhedron is attained at extreme points, we have that for any $s \in \mathcal{S}$ such that $\min_{g \in B_+} \mathbb{E}_g[s] = I$, it holds

$$\min_{g \in B_+} \mathbb{E}_g[s] = \mathbb{E}_{g^i}[s] \tag{22}$$

for some $i \in \{1, \ldots I\}$. This implies that the program 8 for finding $S^*(f)$ can be rewritten as follows:

$$\min_{s \in \mathcal{S}} \left\{ \begin{array}{l} \mathbb{E}_f[s] - \min_{g \in B_+} \mathbb{E}_g[s] \\ \text{s.t.} \quad \min_{g \in B_+} \mathbb{E}_g[s] = I \end{array} \right\} = \min_{s \in \mathcal{S}} \left\{ \begin{array}{l} \mathbb{E}_f[s] - \min_{i \in \{1, \ldots, I\}} \mathbb{E}_{g^i}[s] \\ \text{s.t.} \quad \min_{i \in \{1, \ldots, I\}} \mathbb{E}_{g^i}[s] = I \end{array} \right\}.$$

Let us construct a correspondence, $\mathcal{S}^i$, that maps any $i \in \{1, \ldots I\}$ into the set of types $\mathcal{S}^i \subseteq \mathcal{S}$ such that 1) for any $s \in \mathcal{S}^i$, (22) holds; 2) each $s \in \mathcal{S}$ such that $\min_{g \in B_+} \mathbb{E}_g[s] = I$ belongs to some set $\mathcal{S}^i$, and only one such set.[21] Then we can equivalently find $S^*(f)$ in two steps as follows. In the first step, for each $i = 1, \ldots, I$, we solve

$$S^i \equiv \arg\min_{s \in \mathcal{S}^i} \{\mathbb{E}_f[s] - \mathbb{E}_{g^i}[s]\}. \tag{23}$$

Set $\mathcal{S}^i$ is a convex polyhedron defined by the finite set of inequalities and so, it also has a finite number of extreme points $\bar{S}^i \equiv \{s^1, \ldots, s^{J_i}\} \subset \mathcal{S}^i$. Moreover, the minimized function in (23) is linear in s. Therefore, unless $f = g^i$, the solution to (23) is attained at one of extreme points $\bar{S}^i$ of $\mathcal{S}^i$, i.e., $S^i \subseteq \bar{S}^i$.

In the second step, the issuer type f chooses all indices i to minimize (23). Call this set of indices $I(f) \subseteq \{1, \ldots, I\}$. Then

$$S^*(f) = \cup_{i \in I(f)} S^i \subseteq \cup_{i \in \{1, \ldots, I\}} \bar{S}^i$$

We conclude that since each $\bar{S}^i$ is finite and I is finite, $S^*(f)$ is a finite set for any $f \notin G$.

Consider the set of types B_0 given by the intersection of the hyperplane $\mathbb{E}_f[z] = K$ and B. If s and s' belong to $S^*(f)$, then type f is indifferent between them and so, $\mathbb{E}_f[s - s'] = 0$. Since vectors z and $s - s'$ are not co-linear, the set of types in B_0 who are indifferent between securities s and s' has dimension $N - 2$. Since s and s' belong to $\cup_{i \in \{1, \ldots, I\}} \bar{S}^i$, the set of all types in B_0, who are indifferent between issuing several securities, is a finite union of sets of dimensionality $N - 2$, and hence, itself has dimensionality $N - 2$. This completes the proof of Claim 1. □

[21] This correspondence can be easily constructed inductively. For $i = 1$, let $\mathcal{S}^1$ be the set of all securities in $\mathcal{S}$ such that $\min_{g \in B_+} \mathbb{E}_g[s] = \mathbb{E}_{g^1}[s]$. For $i = 2$, let $\mathcal{S}^2$ be the set of all securities in $\mathcal{S} \backslash \mathcal{S}^1$ such that $\min_{g \in B_+} \mathbb{E}_g[s] = \mathbb{E}_{g^2}[s]$, and so on until we reach $i = I$.

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B Online Appendix (Not for Publication)

B.1 Proofs for Section 3

Proof of Lemma 1. Case 1: $s \in \mathcal{S}^*$. First, consider $s \in \mathcal{S}^*$ satisfying $P(s) > I$. Because $\mathbb{E}_f[z - s] - W = U^*(f)$ for any $f : s^*(f) = s$, set ((5)) is non-empty. Hence, for any securityγs with $\gamma \in (0,1)$, set ((5)) is also non-empty. Hence, $P(\gamma s) = \min_{f \in B(\gamma s)} \mathbb{E}_f[\gamma s]$, where $B(\gamma s)$ is given by ((5)). Let us show that equivalently $P(\gamma s)$ solves

$$\min_{f \in B} \mathbb{E}_f[\gamma s] \tag{24}$$

$$\text{s.t. } \mathbb{E}_f[z - \gamma s] \geq W, \tag{25}$$

$$\mathbb{E}_f[\gamma s] \leq \mathbb{E}_f[\tilde{s}], \forall \tilde{s} \in \mathcal{S}^*. \tag{26}$$

Let $P_1(\gamma s)$ denote the solution to (24)-(26). We want to show that $P_1(\gamma s) = P(\gamma s)$. First, we argue that $f \in B(\gamma s)$, given by ((5)), implies conditions (25)-(26). It cannot be that $\mathbb{E}_f[z - \gamma s] \geq W + U^*(f)$ but $\mathbb{E}_f[z - \gamma s] < W$, because $U^*(f) \geq 0$: No type f could get negative net expected payoff in equilibrium, because she can ensure getting zero by offering **0** and not doing the project. It cannot be that $\mathbb{E}_f[z - \gamma s] \geq W + U^*(f)$ but $\mathbb{E}_f[\gamma s] > \mathbb{E}_f[\tilde{s}]$ for some $\tilde{s} \in \mathcal{S}^*$, because type f would be better off deviating to offering $\tilde{s}$. Indeed, because $\tilde{s} \in \mathcal{S}^*$, such offer would be accepted by the investor and would yield the issuer a strictly higher payoff by $\mathbb{E}_f[s] > \mathbb{E}_f[\gamma s] > \mathbb{E}_f[\tilde{s}]$. Therefore, set $B(s)$ is a subset of $f \in B$ satisfying (25)-(26). Therefore, $P_1(\gamma s) \leq P(\gamma s)$. By contradiction, suppose that $P_1(\gamma s) < P(\gamma s)$ for some γ and s. Then, there is $f' \in B$, satisfying (24)-(26), for which $f' \notin B(\gamma s)$ and $\mathbb{E}_{f'}[s] < \min_{f \in B(\gamma s)} \mathbb{E}_f[s]$. Since set ((5)) is non-empty for s, it is non-empty for γs. Therefore, $f' \notin B(\gamma s)$ implies $\mathbb{E}_{f'}[\gamma s] > \mathbb{E}_{f'}[s']$, where s' is the equilibrium security issued by type f'. However, this inequality contradicts (26) for $\tilde{s} = s'$. Therefore, $P(\gamma s) = P_1(\gamma s)$.

The objective function (24) is continuous in γ and f . The set of constraints, (25)-(26), is continuous in γ and compact for all γ. By the maximum theorem, $P_1(\gamma s)$ is continuous in γ. Since $P(\gamma s) = P_1(\gamma s)$, $P(\gamma s)$ is continuous in γ. Since $P(s) > 1$ and $P(0) = 0$, by continuity there exists $\gamma \in (0,1)$ such that $P(\gamma s) = I$.

Case 2: $s \notin \mathcal{S}^*$. Second, consider $s \notin \mathcal{S}^*$ satisfying $P(s) > I$. Two cases are possible: (1) set ((5)) is non-empty; (2) set ((5)) is empty. If set ((5)) is non-empty, then the proof coincides with the proof of case $s \in \mathcal{S}^*$. If set ((5)) is empty, then $P(s) = \min_{f \in B} \mathbb{E}_f[s]$. Consider set $\{f \in B : \mathbb{E}_f[z - \gamma s] - W \geq U^*(f)\}$ for a fixed s while varying γ. This set is empty for $\gamma = 1$. It is non-empty for $\gamma = 0$, because $\{f \in B : \mathbb{E}_f[z] \geq W + I\} \neq \emptyset$. Therefore, by continuity of $\{f \in B : \mathbb{E}_f[z - \gamma s] - W \geq U^*(f)\}$ in γ, it is empty if and only if$\gamma \geq \gamma^*$ for some $\gamma^* \in (0,1)$. Consider the valuation of security γs in the range $\gamma \in (\gamma^*, 1)$. In this case, $B(\gamma s) = B$, so

$P(\gamma s) = \min_{f\in B}\mathbb{E}_f[\gamma s] = \gamma P(s)$. Clearly, it is continuous in $\gamma \in (\gamma^*, 1)$. If $\gamma^* P(s) < 1$, then $\gamma = \frac{1}{P(s)} \in (\gamma^*, 1)$ yields $P(\gamma s) = I$. If $\gamma^* P(s) \geq 1$, then $P(\gamma^* s) = \min_{f\in B(\gamma^* s)}\mathbb{E}_f[s] \geq I$, and for any $\gamma \in (0, \gamma^*)$, $P(\gamma s) = \min_{f\in B(\gamma s)}\mathbb{E}_f[\gamma s]$ with $B(\gamma s)$ given by ((5)). Since set ((5)) is non-empty, the proof of case $s \in \mathcal{S}^*$ applies. Hence, there exists $\gamma \in (0, \gamma^*)$ satisfying $P(\gamma s) = I$.

By the optimality of equilibrium strategy, it is necessary $P(s) = I$ for any $s \in \mathcal{S}^*$. □

B.2 Proofs for Section 5

Proof of Lemma 5. We first show to preliminary observations. First, by the analogous argument as in Lemma 1, one can show that for any s such that $P(s) > K$, there exists $\gamma \in (0, 1)$ such that $P(\gamma s) = K$. In particular,

$$P(s) = K \text{ for any } s \in S^*. \tag{27}$$

Second, since $\eta \geq K$, type $\underline{\hat{f}}$ indeed prefers to issue some security on the equilibrium path. Denote this security by $s^\dagger$.

Fix an equilibrium. Consider first securities that could be issued in equilibrium, i.e., $P(s) \geq K$. Suppose to contradiction that for some such security s it holds that $\mathbb{E}_{\hat{f}(s)}[s] > \mathbb{E}_{\underline{\hat{f}}}[s]$. If type $\underline{\hat{f}}$ deviates to s, he gets

$$\mathbb{E}_{\underline{\hat{f}}}[\hat{z} - s] > \mathbb{E}_{\underline{\hat{f}}}[\hat{z}] - \mathbb{E}_{\hat{f}(s)}[s] = \mathbb{E}_{\underline{\hat{f}}}[\hat{z}] - K,$$

while in equilibrium, type $\underline{\hat{f}}$ issues some $s^\dagger$ and gets

$$\mathbb{E}_{\underline{\hat{f}}}[\hat{z} - s^\dagger] = \mathbb{E}_{\underline{\hat{f}}}[\hat{z}] - \min_{\hat{f}\in\hat{B}(s^\dagger)}\mathbb{E}_{\hat{f}}[s^\dagger] = \mathbb{E}_{\underline{\hat{f}}}[\hat{z}] - K,$$

where the last equality is by (27). Thus, deviation to s is profitable for type $\underline{\hat{f}}$, which is a contradiction.

Now, consider a security such that $P(s) < K$, i.e., security s could not be issued in equilibrium, and suppose that $\mathbb{E}_{\hat{f}(s)}[s] > \mathbb{E}_{\underline{\hat{f}}}[s]$. The latter implies that $\hat{B}(s)$ does not contain $\hat{f}(s)$. Then, by the definition of $\hat{B}(\cdot)$, $\mathbb{E}_{\underline{\hat{f}}}[s^\dagger] < \mathbb{E}_{\underline{\hat{f}}}[s]$ and for $\tilde{f} \equiv \hat{f}(s) \in B\backslash\underline{\hat{f}}$, $\mathbb{E}_{\tilde{f}}[s] \leq \mathbb{E}_{\tilde{f}}[s^*(\tilde{f})]$. The former inequality implies that $\mathbb{E}_{\tilde{f}}[s^\dagger] < \mathbb{E}_{\tilde{f}}[s]$ and so, $\mathbb{E}_{\tilde{f}}[s^\dagger] < \mathbb{E}_{\tilde{f}}[s^*(\tilde{f})]$, which contradicts the fact that $s^*(\tilde{f})$ is the equilibrium offer of type $\tilde{f}$. Therefore, we conclude that $P(s) = \mathbb{E}_{\underline{\hat{f}}}[s]$ and $\hat{f}(s) = \underline{\hat{f}}$, for any $s \in \mathcal{S}$. □

B.3 Proofs for Section 6

Proof of Lemma 6. 1) Since $s_0 = \tilde{s}_0 = 0$, $\mathbb{E}_f[s - \tilde{s}] = \sum_{n=1}^N f_n(s_n - \tilde{s}_n)$ and so, $\mathbb{E}_f[s - \tilde{s}] \leq 0$ implies $\mathbb{E}_{\gamma f}[s - \tilde{s}] \leq 0$ for any $\gamma > 0$.

2) $\mathbb{E}_{f''}[s - \tilde{s}] = \mathbb{E}_{\alpha f}[s - \tilde{s}] + \mathbb{E}_{(1-\alpha)f'}[s - \tilde{s}] = \alpha\mathbb{E}_f[s - \tilde{s}] + (1 - \alpha)\mathbb{E}_{f'}[s - \tilde{s}] \leq 0$ whenever

$\mathbb{E}_f[s-\tilde{s}] \leq 0$ and $\mathbb{E}_{f'}[s-\tilde{s}] \leq 0$.

3) The first two statements imply that the set of types who weakly prefer to issue s to any $s^* \in S^*$ is a convex cone, which we denote by $C(s)$, and so, $B(s) = C(s) \cap B_+$ by the definition of $B(s)$. □

Proof of Lemma 7. By Lemma 6, it is sufficient to prove the statement for $f \in \mathcal{E} \cap B_0 = \text{relint}(B_0)$. By Lemma 2, for type $f \in \text{relint}(B_0)$, the mispricing from issuing equity $s = \frac{I}{K}z$ equals $\mathbb{E}_f[s] - I = 0$. Again, by Lemma 2, the mispricing from any other security s with $V(s) = 0$ equals $\mathbb{E}_f[s] - \min_{f\in B_+}\mathbb{E}_f[s] \geq 0$. Moreover, since $f \in \text{relint}(B_0)$ and $(s_1,\ldots,s_N)$ is not co-linear to $(z_1,\ldots,z_N)$, $\mathbb{E}_f[s] - \min_{f\in B_+}\mathbb{E}_f[s] > 0$. Therefore, type f strictly prefers to issue equity. □

Proof of Proposition 7. 1) We first prove the following claim:

Claim 2. *There exists a unique $f^* \in B^\nu$ such that $\overline{K} = \mathbb{E}_{f^*}[z]$ and f^* is given by $F^*(z_n) = \max\{\lim_{z\uparrow z_{n+1}} \mathbf{F}(z-\nu) - \nu, 0\}$ for $n < N$ and $F^*(z_N) = 1$.*

Proof: We show that for any $f \in B^\nu$, $F(z) \geq F^*(z)$ for all $z \in Z$. Suppose to contradiction that for some $f \in B^\nu$ and $z_n \in Z$, $F(z_n) < F^*(z_n)$. Then

$$\begin{aligned}
\mathbb{P}_f(z \geq z_{n+1}) &= \mathbb{P}_f(z > z_n) \\
&= 1 - F(z_n) \\
&> 1 - F^*(z_n) \\
&= \min\{1 - \lim_{z\uparrow z_{n+1}} \mathbf{F}(z-\nu) + \nu, 1\} \\
&= \min\{\mathbb{P}_{\mathbf{f}}(z \geq z_{n+1} - \nu) + \nu, 1\},
\end{aligned}$$

which contradicts (1). Thus, for any $f \in B^\nu$, $F(z) \geq F^*(z)$ for all $z \in Z$, and $F(0) > F^*(0)$. Hence, f^* first-order stochastically dominates any $f \in B^\nu$, and so $\mathbb{E}_f[z] < \mathbb{E}_{f^*}[z]$ for all $f \in B\backslash f^*$. *q.e.d.*

Claim 3. *There exists $\gamma^* < 1$ such that $\gamma^* f^*$ is in the interior of B^ν.*

Proof: We will show that for any $A \in 2^Z$, there exists $\gamma < 1$ such that $\mathbb{P}_{\gamma f^*}(A) < \mathbb{P}_{\mathbf{f}}(A^\nu) + \nu$. Since there is a finite number of sets in 2^Z, we can find $\gamma^* < 1$ such that $\mathbb{P}_{\gamma^* f^*}(A) < \mathbb{P}_{\mathbf{f}}(A^\nu) + \nu$ for any $A \in 2^Z$.

Fix $A \in 2^Z$. There are two cases to consider:

1. Suppose $z_0 \notin A$. Then $\mathbb{P}_{\gamma f^*}(A) = \gamma \mathbb{P}_{f^*}(A) < \mathbb{P}_{f^*}(A) \leq \mathbb{P}_{\mathbf{f}}(A^\nu) + \nu$.

2. Suppose $z_0 \in A \neq Z$. (If $A = Z$, the result is trivial). Then

$$\begin{aligned}
\mathbb{P}_{\gamma f^*}(A) &= \gamma \sum_{n: z_n \in A\backslash z_0} f^*_n + 1 - \gamma(1 - f^*_0) \\
&= \gamma \mathbb{P}_{f^*}(A) + 1 - \gamma.
\end{aligned}$$

If we show that $\mathbb{P}_{f^*}(A) < \mathbb{P}_{\mathbf{f}}(A^\nu) + \nu$, then for γ sufficiently close to 1, $\gamma \mathbb{P}_{f^*}(A) + 1 - \gamma < \mathbb{P}_{\mathbf{f}}(A^\nu) + \nu$ and so, $\mathbb{P}_{\gamma f^*}(A) < \mathbb{P}_{\mathbf{f}}(A^\nu) + \nu$.

We partition the set A into disjoint subsets $A_k, k = 1, \dots, K$ as follows:

(a) for any $k = 1, \dots, K$ and any $z, z' \in A_k$ such that $(z, z') \cap A = \emptyset$ it holds $|z - z'| < \nu$;

(b) for any $k = 2, \dots, K$, $\min A_k - \max A_{k-1} > \nu$.

In words, we partition all elements of A into K subsets such that the distance between two consecutive elements of A in each subset A_k is less than ν, while the distance between consecutive subsets is greater than ν. Denote $z_{n_k} = \min A_k$ and $z_{m_k} = \max A_k$. Let k_0 the largest index k such that $F^*(z_{n_k}) = 0$. (If no such index exists, then $k_0 = 1$. In this case, since $z_0 \in A$, $z_{n_0} = z_0$). We suppose that $F^*(z_{m_K}) > 0$ to rule out uninteresting case when $\mathbb{P}_{f^*}(A) = 0$. There are two cases to consider:

(a) Suppose that $z_N \notin A$

$$
\begin{aligned}
\mathbb{P}_{f^*}(A) &= \sum_{k=1}^{K} \mathbb{P}_{f^*}(A_k) \\
&= \sum_{k=k_0+1}^{K} (F^*(z_{m_k}) - F^*(z_{n_k})) + F^*(z_{m_{k_0}}) \\
&= \sum_{k=k_0+1}^{K} (\lim_{z \uparrow z_{m_k+1}} \mathbf{F}(z-\nu) - \lim_{z \uparrow z_{n_k+1}} \mathbf{F}(z-\nu)) + \max\{\lim_{z \uparrow z_{m_{k_0}+1}} \mathbf{F}(z-\nu) - \nu, 0\} \\
&\leq \sum_{k=k_0+1}^{K} (\mathbf{F}(z_{m_k}+\nu) - \lim_{z \uparrow z_{n_k+1}} \mathbf{F}(z-\nu)) + \max\{\lim_{z \uparrow z_{m_{k_0}+1}} \mathbf{F}(z-\nu) - \nu, 0\} \\
&\leq \sum_{k=k_0+1}^{K} (\mathbf{F}(z_{m_k}+\nu) - \lim_{z \uparrow z_{n_k}} \mathbf{F}(z-\nu)) + \max\{\lim_{z \uparrow z_{m_{k_0}+1}} \mathbf{F}(z-\nu) - \nu, 0\} \\
&\leq \sum_{k=k_0+1}^{K} (\mathbf{F}(z_{m_k}+\nu) - \lim_{z \uparrow z_{n_k}} \mathbf{F}(z-\nu)) + \max\{\mathbf{F}(z_{m_{k_0}}) - \nu, 0\} \\
&\leq \sum_{k=k_0+1}^{K} (\mathbf{F}(z_{m_k}+\nu) - \lim_{z \uparrow z_{n_k}} \mathbf{F}(z-\nu)) + \max\{\mathbf{F}(z_{m_{k_0}}) - \nu, 0\} \\
&= \mathbb{P}_{\mathbf{f}}(A^\nu \cap [z_{n_{k_0+1}} - \nu, z_{m_K} + \nu]) + \max\{\mathbf{F}(z_{m_{k_0}}) - \nu, 0\} \\
&< \mathbb{P}_{\mathbf{f}}(A^\nu \cap [z_{n_{k_0+1}} - \nu, z_{m_K} + \nu]) + \mathbb{P}_{\mathbf{f}}(A^\nu \cap [0, z_{m_{k_0}} + \nu]) \\
&\leq \mathbb{P}_{\mathbf{f}}(A^\nu),
\end{aligned}
$$

where the first equality follows from the fact that A_k are disjoint and $A = \cup_{k=1}^{K} A_k$; the second equality is by the definition of k_0 and the fact that $F^*(z_{n_{k_0}}) = 0$; the third equality is by the definition of F^*; the first inequality is by the fact that $\mathbf{F}$ is right-continuous and increasing; the second inequality is by $z_{n_k} < z_{n_{k+1}}$; the third inequality is by the fact that if $z_{m_{k_0}+1} - \nu > z_{m_{k_0}}$, then $\mathbf{F}(z_{m_{k_0}}) = \mathbf{F}(z_{m_{k_0}+1} - \nu)$, and if $z_{m_{k_0}+1} - \nu \leq z_{m_{k_0}}$, then $\mathbf{F}(z_{m_{k_0}}) \geq \mathbf{F}(z_{m_{k_0}+1} - \nu)$; the forth equality is by the fact that $A^\nu \cap [z_{n_{k_0+1}} - \nu, z_{m_K} + \nu] \subseteq \cup_{k=k_0+1}^{K} [z_{n_k} - \nu, z_{m_k} + \nu]$ and the sets $\{[z_{n_k} - \nu, z_{m_k} + \nu]\}_{k=1}^{K}$ are disjoint; the last inequality is by the fact that $[0, z_{m_{k_0}} + \nu]$ and $[z_{n_{k_0+1}} - \nu, z_{m_K} + \nu]$ are disjoint. It remains to show the strict inequality, or equivalently, that

$$\max\{\mathbf{F}(z_{m_{k_0}}) - \nu, 0\} < \mathbb{P}_{\mathbf{f}}(A^\nu \cap [0, z_{m_{k_0}} + \nu]).$$

The inequality holds if $\mathbf{F}(z_{m_{k_0}}) - \nu \leq 0$, as $z_0 \in A$ and so, $\mathbb{P}_{\mathbf{f}}(A^\nu \cap [0, z_{m_{k_0}} + \nu]) > 0$. If $\mathbf{F}(z_{m_{k_0}}) - \nu > 0$, then there are two cases to consider. First, if $z_{n_{k_0}} > z_0$, then

$$\begin{aligned}
\max\{\mathbf{F}(z_{m_{k_0}}) - \nu, 0\} &= \mathbf{F}(z_{m_{k_0}}) - \nu \\
&\leq \mathbf{F}(z_{m_{k_0}}) - \mathbf{F}(z_{n_{k_0}}) \\
&\leq \mathbb{P}_{\mathbf{f}}(A^\nu \cap [z_{n_{k_0}} - \nu, z_{m_{k_0}} + \nu]) \\
&< \mathbb{P}_{\mathbf{f}}(A^\nu \cap [z_{n_{k_0}} - \nu, z_{m_{k_0}} + \nu]) + \mathbb{P}_{\mathbf{f}}(A^\nu \cap [0, z_{m_{k_0-1}} + \nu]) \\
&= \mathbb{P}_{\mathbf{f}}(A^\nu \cap [0, z_{m_{k_0}} + \nu]),
\end{aligned}$$

where the strict inequality is by the fact that $z_0 \in A$. Second, if $z_{n_{k_0}} = z_0$, then

$$\begin{aligned}
\max\{\mathbf{F}(z_{m_{k_0}}) - \nu, 0\} &= \mathbf{F}(z_{m_{k_0}}) - \nu \\
&\leq \mathbf{F}(z_{m_{k_0}} + \nu) - \nu \\
&= \mathbb{P}_{\mathbf{f}}(A^\nu \cap [0, z_{m_{k_0}} + \nu]) - \nu \\
&< \mathbb{P}_{\mathbf{f}}(A^\nu \cap [0, z_{m_{k_0}} + \nu]).
\end{aligned}$$

Therefore, we have shown that $\mathbb{P}_{f^*}(A)$ is strictly less than $\mathbb{P}_{\mathbf{f}}(A^\nu)$ and so, less than $\mathbb{P}_{\mathbf{f}}(A^\nu) + \nu$.

(b) Suppose that $z_N \in A$. Then by the same argument as before

$$\begin{aligned}
\mathbb{P}_{f^*}(A) &= \sum_{k=1}^{K} \mathbb{P}_{f^*}(A_k) \\
&= 1 - F^*(z_{n_K}) + \sum_{k=k_0+1}^{K-1} (F^*(z_{m_k}) - F^*(z_{n_k})) + F^*(z_{m_{k_0}}) \\
&= 1 - \lim_{z \uparrow z_{n_K+1}} \mathbf{F}(z-\nu) + \nu + \sum_{k=k_0+1}^{K-1} (\lim_{z \uparrow z_{m_k+1}} \mathbf{F}(z-\nu) - \lim_{z \uparrow z_{n_k+1}} \mathbf{F}(z-\nu)) \\
&\quad + \max\{\lim_{z \uparrow z_{m_{k_0}+1}} \mathbf{F}(z-\nu) - \nu, 0\} \\
&\leq 1 - \lim_{z \uparrow z_{n_K+1}} \mathbf{F}(z-\nu) + \nu + \sum_{k=k_0+1}^{K} (\mathbf{F}(z_{m_k}+\nu) - \lim_{z \uparrow z_{n_k}} \mathbf{F}(z-\nu)) + \max\{\mathbf{F}(z_{m_{k_0}}) - \nu, 0\} \\
&= \nu + \mathbb{P}_{\mathbf{f}}(A^\nu \cap [z_{n_{k_0+1}} - \nu, z_N]) + \max\{\mathbf{F}(z_{m_{k_0}}) - \nu, 0\} \\
&< \nu + \mathbb{P}_{\mathbf{f}}(A^\nu \cap [z_{n_{k_0+1}} - \nu, z_N]) + \mathbb{P}_{\mathbf{f}}(A^\nu \cap [0, z_{m_{k_0}} + \nu]) \\
&\leq \nu + \mathbb{P}_{\mathbf{f}}(A^\nu).
\end{aligned}$$

q.e.d.

We can now prove the first statement of the theorem. We will show that there exists $\underline{K} < \overline{K}$ such that for $K \in [\underline{K}, \overline{K}]$ it holds that for any $f \in B_+^{\nu,K}$, there exist $\gamma \leq 1$ and $\psi \in B_0^{\nu,K}$ such that $f = \frac{1}{\gamma}\psi$. Suppose to contradiction that for any $K < \overline{K}$, there is f_K such that $\{\gamma f_K, \gamma \leq 1\} \cap B_0^{\nu,K} = \emptyset$. This implies that $\{\gamma f_K, \gamma \leq 1\} \cap \{f \in B^\nu : \mathbb{E}_f[z] \leq K\} = \emptyset$. By Claim 2, f^* is the unique limit point of the sequence f_K as $K \to \overline{K}$. By the construction of f_K, for any $\gamma < 1$, γf^* is a boundary point of B^ν, which contradicts Claim 3. Therefore, there exists $\underline{K} < \overline{K}$ such that for any $K \in [\underline{K}, \overline{K}]$ it holds that for any $f \in B_+^{\nu,K}$, there exist $\gamma \leq 1$ and $\psi \in B_0^{\nu,K}$ such that $f = \frac{1}{\gamma}\psi$. By Lemma 7, all types in the interior of $B_+^{\nu,K}$ issue equity, which completes the proof of the first statement.

2) As ν increases, the set $B_0^{\nu,K}$ converges in the Hausdorff metric to the set $\{f \in \Delta(Z) : \mathbb{E}_f[z] = K\}$, and so, cone $\mathcal{E}^{\nu,K}$ converges to $\{f \in \Delta(Z) : \mathbb{E}_f[z] \geq K\}$. The second statement then follows from the fact that $B_+^{\nu,K}$ also converges to $\{f \in \Delta(Z) : \mathbb{E}_f[z] \geq K\}$. □

Proof of Lemma 8. We show that for any $f \in B$, $F(z) \leq \underline{F}(z)$ for all $z \in Z$. Suppose to contradiction that for some $f \in B$ and $z_n \in Z$, $F(z_n) > F^*(z_n)$. Then

$$\begin{aligned}\mathbb{P}_f(z \leq z_n) &= F(z_n) \\ &> \underline{F}(z_n) \\ &= \min\{\mathbf{F}(z_n + \nu) + \nu, 1\} \\ &= \min\{\mathbb{P}_{\mathbf{f}}(z \leq z_n + \nu) + \nu, 1\}\end{aligned}$$

which contradicts (1). Thus, for any $f \in B$, $F(z) \leq \underline{F}(z)$ for all $z \in Z$, and $F(0) < \underline{F}(0)$. Hence, any $f \in B$ first-order stochastically dominates $\underline{f}$, and so for any $s \in \mathcal{S}$, $\mathbb{E}_f[s] > \mathbb{E}_{\underline{f}}[s]$. □

B.4 Proofs for Section 7

The analysis of the Euclidean model proceeds in a series of lemmas.

Lemma 10. *There is a maximal set $[\underline{\alpha}, \overline{\alpha}] \subseteq [0, 1]$ such that $s^\alpha \in \mathcal{S}$ for all $\alpha \in [\underline{\alpha}, \overline{\alpha}]$.*

Proof. We need to show that if $s^{\underline{\alpha}}, s^{\overline{\alpha}} \in \mathcal{S}$, then $s^\alpha \in \mathcal{S}$ for any $\alpha \in (\underline{\alpha}, \overline{\alpha})$ (the fact that the maximal set is closed follows from the continuity argument). Consider

$$\tilde{\psi} = \{f : \mathbb{E}_f[s^{\underline{\alpha}}] = I\} \cap \{f : \mathbb{E}_f[s^{\overline{\alpha}}] = I\}.$$

Choose $\gamma \in (0, 1)$ such that $\alpha = \gamma\underline{\alpha} + (1 - \gamma)\overline{\alpha}$ and let $\tilde{s} = \gamma s^{\underline{\alpha}} + (1 - \gamma)s^{\overline{\alpha}} \in \mathcal{S}$. Then $\mathbb{E}_{\tilde{\psi}}[\tilde{s}] = I$. By the strict convexity of B and the definition of $\psi^{\underline{\alpha}}$ and $\psi^{\overline{\alpha}}$, $\tilde{\psi}$ does not belong to the interior of B. This implies that $\mathbb{E}_{\psi^\alpha}[\tilde{s}] > I$ and so, $s^\alpha = \delta\tilde{s}$ for some $\delta \in (0, 1]$. Since $\tilde{s} \in \mathcal{S}$, $s^\alpha = \delta\tilde{s} \in \mathcal{S}$, which gives the desired conclusion. □

Lemma 11. *For any α and α' in $[\underline{\alpha}, \overline{\alpha}]$, if $f = \gamma\psi^\alpha$ for some $\gamma \geq 1$, then $\mathbb{E}_f[s^\alpha] < \mathbb{E}_f[s^{\alpha'}]$.*

Proof. It is sufficient to show that $\mathbb{E}_{\psi^\alpha}[s^\alpha] < \mathbb{E}_{\psi^\alpha}[s^{\alpha'}]$, as it is equivalent to $\mathbb{E}_f[s^\alpha] < \mathbb{E}_f[s^{\alpha'}]$. Using the definitions of ψ^α and s^α, and the strict convexity of B, we get

$$\begin{aligned}\mathbb{E}_{\psi^\alpha}[s^\alpha] &= I \\ &= \mathbb{E}_{\psi^{\alpha'}}[s^{\alpha'}] \\ &= \min_{f \in B} \mathbb{E}_f[s^{\alpha'}] \\ &< \mathbb{E}_{\psi^\alpha}[s^{\alpha'}],\end{aligned}$$

which is the desired conclusion. □

Proof of Proposition 9. Lemma 11 implies that whenever type $f = \gamma\psi^{\alpha}$ can issue security s^{α}, he would prefer it to any other security $s^{\alpha'}, \alpha' \in [\underline{\alpha}, \overline{\alpha}]$. This is the case in the no adverse selection case. The optimality of $s^{\overline{\alpha}}$ for types with $\zeta_f \geq \zeta_{\psi^{\overline{\alpha}}}$ and call option $s^{\underline{\alpha}}$ for types with $\zeta_f \leq \zeta_{\psi^{\underline{\alpha}}}$ follows from the same argument as in the proof of Proposition 2.

Finally, we show that $s^{\overline{\alpha}}$ is the risky debt, and by the analogous argument, $s^{\underline{\alpha}}$ is the call option. Suppose to contradiction that security $s^{\overline{\alpha}}$ is not the risky debt, that is, $s_1^{\overline{\alpha}} < \min\{z_1, s_2^{\overline{\alpha}}\}$. This implies that there exists a security $s \in \mathcal{S}$ such that $s_1 > s_1^{\overline{\alpha}}$ and $s_2 < s_2^{\overline{\alpha}}$, and $\mathbb{E}_{\psi^{\alpha}}[s] > I$. This, in turn, implies that for $\alpha = \frac{s_1}{s_2} > \overline{\alpha}$, there exists $s^{\alpha} \in \mathcal{S}$ such that $\mathbb{E}_{\psi^{\alpha}}[s^{\alpha}] = I$, which contradicts the definition of $\overline{\alpha}$. Therefore, $s_1^{\overline{\alpha}} = \min\{z_1, s_2^{\overline{\alpha}}\}$, and $s^{\overline{\alpha}}$ is the risky debt. □

Proof of Proposition 10. Let us define for any $\alpha \in [0, 1]$,

$$\tilde{\psi}^{\alpha} \equiv \arg\min_{f \in B_+} \{\alpha f_1 + f_2\},$$

and let $\tilde{s}^{\alpha}$ be the security such that $\mathbb{E}_{\tilde{\psi}^{\alpha}}[\tilde{s}^{\alpha}] = I$ and $\frac{\tilde{s}_1^{\alpha}}{\tilde{s}_2^{\alpha}} = \alpha$, whenever such a security exists. By the analogous argument to Lemmas 10 and 11, there is a maximal set $[\underline{\alpha}', \overline{\alpha}']$ such that $\tilde{s}^{\alpha} \in \mathcal{S}$ for all $\alpha \in [\underline{\alpha}', \overline{\alpha}']$, and types $\{\gamma\tilde{\psi}^{\alpha} : \gamma \geq 1\}$ prefer to issue $\tilde{s}^{\alpha}$ to any other security. Thus, to find the equilibrium security offer for each type f, we need to determine the direction $\tilde{\psi}^{\alpha}$ of the line connecting 0 and f, and find corresponding $\tilde{s}^{\alpha}$. By the same argument as in Propositions 3 and 9, we can verify that the equilibrium offers $s^*(f)$ are as in the statement of the proposition. □

B.5 Proofs for Section 8

Lemma 12. *1. For any s such that $P^{\omega}(s) > I$, there exists $\gamma \in (0, 1)$ such that $P^{\omega}(\gamma s) = I$. In particular, $P^{\omega}(s) = I$ for any $s \in S^*$.*

2. Define $f^{\omega}(s) = \omega f(s) + (1 - \omega)\overline{f}$. For any security $s \in \mathcal{S}$ such that $P^{\omega}(s) = I$, it holds $f^{\omega}(s) \in \arg\min_{f^{\omega} \in B_+^{\omega}} \mathbb{E}_{f^{\omega}}[s]$ and $P^{\omega}(s) = \min_{f^{\omega} \in B_+^{\omega}} \mathbb{E}_{f^{\omega}}[s]$.

Proof. The first part follows by an identical argument to Lemma 1. To prove the second part, we follow the argument analogous to Lemma 3. We first show that $f^{\omega}(s) \in B_+^{\omega}$ whenever $P^{\omega}(s) = I$. By the definition of the model-updating mapping $B(\cdot)$, $f(s) \in B(s)$ implies $\mathbb{E}_{f(s)}[z - s] \geq W$, and so, $\mathbb{E}_{f^{\omega}(s)}[z - s] \geq W$. If $\mathbb{E}_{f^{\omega}(s)}[z] < K$, then

$$\begin{aligned} P^{\omega}(s) &= \mathbb{E}_{f^{\omega}(s)}[s] \\ &= \mathbb{E}_{f^{\omega}(s)}[z] - \mathbb{E}_{f^{\omega}(s)}[z - s] \\ &\leq \mathbb{E}_{f^{\omega}(s)}[z] - W \\ &= \mathbb{E}_{f^{\omega}(s)}[z] - K + I < I, \end{aligned}$$

which contradicts to $P^\omega(s) = I$. Thus, $f^\omega(s) \in B_+^\omega$.

Now, suppose that for some $s \in \mathcal{S}$, it holds that $P^\omega(s) = I$, but $P^\omega(s) = \mathbb{E}_{f^\omega(s)}[s] > \mathbb{E}_{\tilde{f}^\omega}[s]$ for some $\tilde{f}^\omega \in \arg\min_{f^\omega \in B_+^\omega} \mathbb{E}_{f^\omega}[s]$. Observe that $\tilde{f}^\omega \in \arg\min_{f^\omega \in B_+^\omega} \mathbb{E}_{f^\omega}[s]$ implies that corresponding issuer type $\tilde{f} = \frac{\tilde{f}^\omega - (1-\omega)\overline{f}}{\omega}$ issues some security $\tilde{s} = s^*(\tilde{f})$ in equilibrium. This implies that $\mathbb{E}_{\tilde{f}^\omega}[\tilde{s}] \leq \mathbb{E}_{\tilde{f}^\omega}[s]$ and so,

$$I = P^\omega(\tilde{s}) \leq \mathbb{E}_{\tilde{f}^\omega}[\tilde{s}] \leq \mathbb{E}_{\tilde{f}^\omega}[s] < \mathbb{E}_{f^\omega(s)}[s] = I,$$

which is a contradiction. □